EFDE for D1Q3 with constant velocities,
supplementary material for
Equivalent Finite Difference Equations and Equivalent Partial Differential Equations for the Lattice Boltzmann Method

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1 Global definitions

In $\mathbb{R}$, the position and velocity vectors are given by $\boldsymbol{x} = (x)$ and $\boldsymbol{u} = (u)$, respectively.
Discrete velocity vectors:

$$\{\boldsymbol{c}_i\}_{i=1}^{3} = \left((0)\,,(1)\,,(-1)\right).$$

Equilibrium DF vector $\boldsymbol{f}^{eq}$:

$$\boldsymbol{f}^{eq} = \begin{pmatrix} 1 - c_s^2 - u^2 \\ \frac{1}{2}u + \frac{1}{2}c_s^2 + \frac{1}{2}u^2 \\ -\frac{1}{2}u + \frac{1}{2}c_s^2 + \frac{1}{2}u^2 \end{pmatrix}.$$

Lattice speed of sound: $c_s = \frac{1}{\sqrt{3}}$.
Moments $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)^T$ are given by

$$\boldsymbol{\mu} = \tilde{\mathbf{M}}\boldsymbol{f},$$

where $\boldsymbol{f} = (f_1, f_2, f_3)^T$ and

$$\tilde{\mathbf{M}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

2 SRT

2.1 Definitions

Matrix $\mathbf{A} = \mathbf{S}$:

$$\mathbf{A} = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

where

$$\mathbf{S} = \mathrm{diag}(\omega, \omega, \omega).$$

Matrix $\mathbf{B}$:

$$\mathbf{B} = \begin{pmatrix} 0 & -1+\omega & -1+\omega \\ -1+\omega & 0 & -1+\omega \\ -1+\omega & -1+\omega & 0 \end{pmatrix}.$$

2.2 EFDE for μ_1

$$\boxed{\mu_{1,x}^{t+\delta_t} = \sum_{\ell=0}^{2} \sum_{i=-\ell-1}^{\ell+1} \alpha_{x+i\delta_l}^{[\mu_1],t-\ell\delta_t} \mu_{x+i\delta_l}^{t-\ell\delta_t},}$$

where the non-zero coefficients are given by:
$\alpha_{x-\delta_l}^{[\mu_1],t} = 1 + \frac{1}{2}u\omega - \omega + \frac{1}{2}{c_s}^2\omega + \frac{1}{2}u^2\omega.$

$\alpha_{x}^{[\mu_1],t} = 1 - {c_s}^2\omega - u^2\omega.$

$\alpha_{x+\delta_l}^{[\mu_1],t} = 1 - \frac{1}{2}u\omega - \omega + \frac{1}{2}{c_s}^2\omega + \frac{1}{2}u^2\omega.$

$\alpha_{x-\delta_l}^{[\mu_1],t-\delta_t} = -1 + \frac{1}{2}u\omega^2 - \frac{1}{2}u\omega + \omega + \frac{1}{2}{c_s}^2\omega + \frac{1}{2}u^2\omega - \frac{1}{2}u^2\omega^2 - \frac{1}{2}{c_s}^2\omega^2,$

$\alpha_{x}^{[\mu_1],t-\delta_t} = -1 + 2\omega - {c_s}^2\omega - u^2\omega - \omega^2 + u^2\omega^2 + {c_s}^2\omega^2,$

$\alpha_{x+\delta_l}^{[\mu_1],t-\delta_t} = -1 - \frac{1}{2}u\omega^2 + \frac{1}{2}u\omega + \omega + \frac{1}{2}{c_s}^2\omega + \frac{1}{2}u^2\omega - \frac{1}{2}u^2\omega^2 - \frac{1}{2}{c_s}^2\omega^2,$

$\alpha_{x}^{[\mu_1],t-2\delta_t} = 1 - 2\omega + \omega^2,$

2.3 EFDE for μ_2

$$\boxed{\mu_{2,x}^{t+\delta_t} = \sum_{\ell=0}^{2} \sum_{i=-\ell-1}^{\ell+1} \alpha_{x+i\delta_l}^{[\mu_1],t-\ell\delta_t} \mu_{x+i\delta_l}^{t-\ell\delta_t} + \sum_{\ell=0}^{2} \sum_{i=-\ell-1}^{\ell+1} \alpha_{x+i\delta_l}^{[\mu_2],t-\ell\delta_t} \mu_{2,x+i\delta_l}^{t-\ell\delta_t},}$$

where the non-zero coefficients are given by:

$\alpha_{x-\delta_l}^{[\mu_1],t} = 1 + \frac{1}{2}u\omega - \omega + \frac{1}{2}{c_s}^2\omega + \frac{1}{2}u^2\omega.$

$\alpha_{x+\delta_l}^{[\mu_1],t} = -1 + \frac{1}{2}u\omega + \omega - \frac{1}{2}{c_s}^2\omega - \frac{1}{2}u^2\omega.$

$\alpha_{x-\delta_l}^{[\mu_1],t-\delta_t} = -1 + \omega + {c_s}^2\omega + u^2\omega - u^2\omega^2 - {c_s}^2\omega^2,$

$\alpha_{x-\delta_l}^{[\mu_2],t-\delta_t} = 1 - 2\omega + \omega^2,$

$\alpha_{x}^{[\mu_1],t-\delta_t} = -u\omega^2 + u\omega,$

$\alpha_{x}^{[\mu_2],t-\delta_t} = 1 - 2\omega + \omega^2,$

$\alpha_{x+\delta_l}^{[\mu_1],t-\delta_t} = 1 - \omega - {c_s}^2\omega - u^2\omega + u^2\omega^2 + {c_s}^2\omega^2,$

$\alpha_{x+\delta_l}^{[\mu_2],t-\delta_t} = 1 - 2\omega + \omega^2,$

$\alpha_{x}^{[\mu_1],t-2\delta_t} = -2u\omega^3 + 4u\omega^2 - 2u\omega,$

$\alpha_{x}^{[\mu_2],t-2\delta_t} = -2 + 6\omega - 6\omega^2 + 2\omega^3,$

2.4 EFDE for μ_3

$$\boxed{\mu_{3,x}^{t+\delta_t} = \sum_{\ell=0}^{2} \sum_{i=-\ell-1}^{\ell+1} \alpha_{x+i\delta_l}^{[\mu_1],t-\ell\delta_t} \mu_{x+i\delta_l}^{t-\ell\delta_t} + \sum_{\ell=0}^{2} \sum_{i=-\ell-1}^{\ell+1} \alpha_{x+i\delta_l}^{[\mu_3],t-\ell\delta_t} \mu_{3,x+i\delta_l}^{t-\ell\delta_t},}$$

where the non-zero coefficients are given by:

$\alpha_{x-\delta_l}^{[\mu_1],t} = 1 + \frac{1}{2}u\omega - \omega + \frac{1}{2}{c_s}^2\omega + \frac{1}{2}u^2\omega.$

$\alpha_{x+\delta_l}^{[\mu_1],t} = 1 - \frac{1}{2}u\omega - \omega + \frac{1}{2}{c_s}^2\omega + \frac{1}{2}u^2\omega.$

$\alpha_{x-\delta_l}^{[\mu_1],t-\delta_t} = -1 + \omega + {c_s}^2\omega + u^2\omega - u^2\omega^2 - {c_s}^2\omega^2,$

$\alpha_{x-\delta_l}^{[\mu_3],t-\delta_t} = 1 - 2\omega + \omega^2,$

$$\alpha_{x}^{[\mu_1],t-\delta_t} = -2 + 4\omega - {c_s}^2\omega - u^2\omega - 2\omega^2 + u^2\omega^2 + {c_s}^2\omega^2,$$
$$\alpha_{x}^{[\mu_3],t-\delta_t} = 1 - 2\omega + \omega^2,$$
$$\alpha_{x+\delta_l}^{[\mu_1],t-\delta_t} = -1 + \omega + {c_s}^2\omega + u^2\omega - u^2\omega^2 - {c_s}^2\omega^2,$$
$$\alpha_{x+\delta_l}^{[\mu_3],t-\delta_t} = 1 - 2\omega + \omega^2,$$
$$\alpha_{x}^{[\mu_1],t-2\delta_t} = 2 - 4\omega - 2{c_s}^2\omega - 2u^2\omega + 2\omega^2 - 2{c_s}^2\omega^3 + 4u^2\omega^2 + 4{c_s}^2\omega^2 - 2u^2\omega^3,$$
$$\alpha_{x}^{[\mu_3],t-2\delta_t} = -2 + 6\omega - 6\omega^2 + 2\omega^3,$$

3 MRT

3.1 Definitions

Matrix $\mathbf{A} = \mathbf{M}^{-1}\mathbf{S}\mathbf{M}$:

$$\mathbf{A} = \begin{pmatrix} \omega_0 & \omega_0 - \omega_2 & \omega_0 - \omega_2 \\ 0 & \frac{1}{2}\omega_2 + \frac{1}{2}\omega_1 & \frac{1}{2}\omega_2 - \frac{1}{2}\omega_1 \\ 0 & \frac{1}{2}\omega_2 - \frac{1}{2}\omega_1 & \frac{1}{2}\omega_2 + \frac{1}{2}\omega_1 \end{pmatrix},$$

where

$$\mathbf{S} = \mathrm{diag}(\omega_0, \omega_1, \omega_2)$$

and

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Matrix $\mathbf{B}$:

$$\mathbf{B} = \begin{pmatrix} 0 & -1 + \omega_2 & -1 + \omega_2 \\ -1 + \frac{1}{2}\omega_2 + \frac{1}{2}\omega_1 & 0 & -1 + \omega_1 \\ -1 + \frac{1}{2}\omega_2 + \frac{1}{2}\omega_1 & -1 + \omega_1 & 0 \end{pmatrix}.$$

3.2 EFDE for μ_1

$$\boxed{\mu_{1,x}^{t+\delta_t} = \sum_{\ell=0}^{2} \sum_{i=-\ell-1}^{\ell+1} \alpha_{x+i\delta_l}^{[\mu_1],t-\ell\delta_t} \mu_{x+i\delta_l}^{t-\ell\delta_t},}$$

where the non-zero coefficients are given by:

$$\alpha_{x-\delta_l}^{[\mu_1],t} = 1 + \tfrac{1}{2}\omega_2{c_s}^2 + \tfrac{1}{2}\omega_2 u^2 - \tfrac{1}{2}\omega_2 + \tfrac{1}{2}u\omega_1 - \tfrac{1}{2}\omega_1.$$
$$\alpha_{x}^{[\mu_1],t} = 1 - \omega_2{c_s}^2 - \omega_2 u^2.$$
$$\alpha_{x+\delta_l}^{[\mu_1],t} = 1 + \tfrac{1}{2}\omega_2{c_s}^2 + \tfrac{1}{2}\omega_2 u^2 - \tfrac{1}{2}\omega_2 - \tfrac{1}{2}u\omega_1 - \tfrac{1}{2}\omega_1.$$
$$\alpha_{x-\delta_l}^{[\mu_1],t-\delta_t} = -1 + \tfrac{1}{2}\omega_2{c_s}^2 + \tfrac{1}{2}\omega_2 u^2 + \tfrac{1}{2}\omega_2 - \tfrac{1}{2}\omega_2 u^2\omega_1 - \tfrac{1}{2}u\omega_1 - \tfrac{1}{2}\omega_2{c_s}^2\omega_1 + \tfrac{1}{2}\omega_2 u\omega_1 + \tfrac{1}{2}\omega_1,$$
$$\alpha_{x}^{[\mu_1],t-\delta_t} = -1 - \omega_2{c_s}^2 - \omega_2 u^2 + \omega_2 + \omega_2 u^2\omega_1 - \omega_2\omega_1 + \omega_2{c_s}^2\omega_1 + \omega_1,$$

$\alpha_{x+\delta_l}^{[\mu_1],t-\delta_t} = -1 + \frac{1}{2}\omega_2 {c_s}^2 + \frac{1}{2}\omega_2 u^2 + \frac{1}{2}\omega_2 - \frac{1}{2}\omega_2 u^2 \omega_1 + \frac{1}{2}u\omega_1 - \frac{1}{2}\omega_2 {c_s}^2 \omega_1 - \frac{1}{2}\omega_2 u \omega_1 + \frac{1}{2}\omega_1,$

$\alpha_{x}^{[\mu_1],t-2\delta_t} = 1 - \omega_2 + \omega_2\omega_1 - \omega_1,$

3.3 EFDE for μ_2

$$\boxed{\mu_{2,x}^{t+\delta_t} = \sum_{\ell=0}^{2}\sum_{i=-\ell-1}^{\ell+1} \alpha_{x+i\delta_l}^{[\mu_1],t-\ell\delta_t}\mu_{x+i\delta_l}^{t-\ell\delta_t} + \sum_{\ell=0}^{2}\sum_{i=-\ell-1}^{\ell+1} \alpha_{x+i\delta_l}^{[\mu_2],t-\ell\delta_t}\mu_{2,x+i\delta_l}^{t-\ell\delta_t},}$$

where the non-zero coefficients are given by:

$\alpha_{x-\delta_l}^{[\mu_1],t} = 1 + \frac{1}{2}\omega_2 {c_s}^2 + \frac{1}{2}\omega_2 u^2 - \frac{1}{2}\omega_2 + \frac{1}{2}u\omega_1 - \frac{1}{2}\omega_1.$

$\alpha_{x+\delta_l}^{[\mu_1],t} = -1 - \frac{1}{2}\omega_2 {c_s}^2 - \frac{1}{2}\omega_2 u^2 + \frac{1}{2}\omega_2 + \frac{1}{2}u\omega_1 + \frac{1}{2}\omega_1.$

$\alpha_{x-\delta_l}^{[\mu_1],t-\delta_t} = -1 + \omega_2 {c_s}^2 + \omega_2 u^2 + \frac{1}{2}\omega_2 - \frac{1}{2}\omega_2 u^2\omega_1 - \frac{1}{2}\omega_2 {c_s}^2\omega_1 - \frac{1}{2}\omega_2^2 u^2 - \frac{1}{2}\omega_2^2 {c_s}^2 + \frac{1}{2}\omega_1,$

$\alpha_{x-\delta_l}^{[\mu_2],t-\delta_t} = 1 - \frac{3}{2}\omega_2 + \frac{1}{2}\omega_2^2 + \frac{1}{2}\omega_2\omega_1 - \frac{1}{2}\omega_1,$

$\alpha_{x}^{[\mu_1],t-\delta_t} = -u\omega_1^2 + u\omega_1,$

$\alpha_{x}^{[\mu_2],t-\delta_t} = 1 + \omega_1^2 - 2\omega_1,$

$\alpha_{x+\delta_l}^{[\mu_1],t-\delta_t} = 1 - \omega_2 {c_s}^2 - \omega_2 u^2 - \frac{1}{2}\omega_2 + \frac{1}{2}\omega_2 u^2\omega_1 + \frac{1}{2}\omega_2 {c_s}^2\omega_1 + \frac{1}{2}\omega_2^2 u^2 + \frac{1}{2}\omega_2^2 {c_s}^2 - \frac{1}{2}\omega_1,$

$\alpha_{x+\delta_l}^{[\mu_2],t-\delta_t} = 1 - \frac{3}{2}\omega_2 + \frac{1}{2}\omega_2^2 + \frac{1}{2}\omega_2\omega_1 - \frac{1}{2}\omega_1,$

$\alpha_{x}^{[\mu_1],t-2\delta_t} = u\omega_1^2 - 2u\omega_1 - \omega_2^2 u\omega_1 + 3\omega_2 u\omega_1 - \omega_2 u\omega_1^2,$

$\alpha_{x}^{[\mu_2],t-2\delta_t} = -2 + \omega_2\omega_1^2 - \omega_1^2 + 3\omega_2 - \omega_2^2 - 4\omega_2\omega_1 + \omega_2^2\omega_1 + 3\omega_1,$

3.4 EFDE for μ_3

$$\boxed{\mu_{3,x}^{t+\delta_t} = \sum_{\ell=0}^{2}\sum_{i=-\ell-1}^{\ell+1} \alpha_{x+i\delta_l}^{[\mu_1],t-\ell\delta_t}\mu_{x+i\delta_l}^{t-\ell\delta_t} + \sum_{\ell=0}^{2}\sum_{i=-\ell-1}^{\ell+1} \alpha_{x+i\delta_l}^{[\mu_3],t-\ell\delta_t}\mu_{3,x+i\delta_l}^{t-\ell\delta_t},}$$

where the non-zero coefficients are given by:

$\alpha_{x-\delta_l}^{[\mu_1],t} = 1 + \frac{1}{2}\omega_2 {c_s}^2 + \frac{1}{2}\omega_2 u^2 - \frac{1}{2}\omega_2 + \frac{1}{2}u\omega_1 - \frac{1}{2}\omega_1.$

$\alpha_{x+\delta_l}^{[\mu_1],t} = 1 + \frac{1}{2}\omega_2 {c_s}^2 + \frac{1}{2}\omega_2 u^2 - \frac{1}{2}\omega_2 - \frac{1}{2}u\omega_1 - \frac{1}{2}\omega_1.$

$\alpha_{x-\delta_l}^{[\mu_1],t-\delta_t} = -1 + \omega_2 {c_s}^2 + \omega_2 u^2 + \frac{1}{2}\omega_2 - \frac{1}{2}\omega_2 u^2\omega_1 - \frac{1}{2}\omega_2 {c_s}^2\omega_1 - \frac{1}{2}\omega_2^2 u^2 - \frac{1}{2}\omega_2^2 {c_s}^2 + \frac{1}{2}\omega_1,$

$$\alpha_{x-\delta_l}^{[\mu_3],t-\delta_t} = 1 - \tfrac{3}{2}\omega_2 + \tfrac{1}{2}\omega_2^2 + \tfrac{1}{2}\omega_2\omega_1 - \tfrac{1}{2}\omega_1,$$
$$\alpha_{x}^{[\mu_1],t-\delta_t} = -2 - \omega_2 {c_s}^2 - \omega_1^2 - \omega_2 u^2 + \omega_2 + \omega_2 u^2\omega_1 - \omega_2\omega_1 + \omega_2 {c_s}^2\omega_1 + 3\omega_1,$$
$$\alpha_{x}^{[\mu_3],t-\delta_t} = 1 + \omega_1^2 - 2\omega_1,$$
$$\alpha_{x+\delta_l}^{[\mu_1],t-\delta_t} = -1 + \omega_2 {c_s}^2 + \omega_2 u^2 + \tfrac{1}{2}\omega_2 - \tfrac{1}{2}\omega_2 u^2\omega_1 - \tfrac{1}{2}\omega_2 {c_s}^2\omega_1 - \tfrac{1}{2}\omega_2^2 u^2 - \tfrac{1}{2}\omega_2^2 {c_s}^2 + \tfrac{1}{2}\omega_1,$$
$$\alpha_{x+\delta_l}^{[\mu_3],t-\delta_t} = 1 - \tfrac{3}{2}\omega_2 + \tfrac{1}{2}\omega_2^2 + \tfrac{1}{2}\omega_2\omega_1 - \tfrac{1}{2}\omega_1,$$
$$\alpha_{x}^{[\mu_1],t-2\delta_t} = 2 - 2\omega_2 {c_s}^2 + \omega_1^2 - \omega_2 u^2\omega_1^2 - 2\omega_2 u^2 - \omega_2 - \omega_2^2 {c_s}^2\omega_1 + 3\omega_2 u^2\omega_1 + \omega_2\omega_1 - \omega_2^2 u^2\omega_1 + 3\omega_2 {c_s}^2\omega_1 + \omega_2^2 u^2 + \omega_2^2 {c_s}^2 - 3\omega_1 - \omega_2 {c_s}^2\omega_1^2,$$
$$\alpha_{x}^{[\mu_3],t-2\delta_t} = -2 + \omega_2\omega_1^2 - \omega_1^2 + 3\omega_2 - \omega_2^2 - 4\omega_2\omega_1 + \omega_2^2\omega_1 + 3\omega_1,$$

4 CLBM

4.1 Definitions

Matrix $\mathbf{A} = \mathbf{K}^{-1}\mathbf{S}\mathbf{K}$:

$$\begin{aligned}
\mathbf{A}_{1,1} &= 2\omega_1 u^2 - u^2\omega_2 + \omega_0 - \omega_0 u^2,\\
\mathbf{A}_{1,2} &= (2\omega_1 u - \omega_0 - u(\omega_0 + \omega_2) + \omega_2)(-1 + u),\\
\mathbf{A}_{1,3} &= (2\omega_1 u + \omega_0 - u(\omega_0 + \omega_2) - \omega_2)(1 + u),\\
\mathbf{A}_{2,1} &= -\frac{1}{2}(\omega_1(1 + 2u) - \omega_0 - u(\omega_0 + \omega_2))u,\\
\mathbf{A}_{2,2} &= -\omega_1 u^2 + \frac{1}{2}\omega_1 + \frac{1}{2}\omega_1 u + \frac{1}{2}u^2\omega_2 - u\omega_2 + \frac{1}{2}\omega_0 u^2 + \frac{1}{2}\omega_2 + \frac{1}{2}\omega_0 u,\\
\mathbf{A}_{2,3} &= -\frac{1}{2}(\omega_1(1 + 2u) - u(\omega_0 + \omega_2) - \omega_2)(1 + u),\\
\mathbf{A}_{3,1} &= -\frac{1}{2}(\omega_0 + \omega_1(-1 + 2u) - u(\omega_0 + \omega_2))u,\\
\mathbf{A}_{3,2} &= -\frac{1}{2}(-1 + u)(\omega_1(-1 + 2u) - u(\omega_0 + \omega_2) + \omega_2),\\
\mathbf{A}_{3,3} &= -\omega_1 u^2 + \frac{1}{2}\omega_1 - \frac{1}{2}\omega_1 u + \frac{1}{2}u^2\omega_2 + u\omega_2 + \frac{1}{2}\omega_0 u^2 + \frac{1}{2}\omega_2 - \frac{1}{2}\omega_0 u.
\end{aligned}$$

where

$$\mathbf{S} = \mathrm{diag}(\omega_0, \omega_1, \omega_2)$$

and

$$\mathbf{K} = \begin{pmatrix} 1 & 1 & 1 \\ -u & 1-u & -u-1 \\ u^2 & (1-u)^2 & (u+1)^2 \end{pmatrix}.$$

Matrix $\mathbf{B}$:

$$\mathbf{B} = \begin{pmatrix} 0 & -1 + 2\omega_1 u - 2u\omega_2 + \omega_2 & -1 - 2\omega_1 u + 2u\omega_2 + \omega_2 \\ -1 + \frac{1}{2}\omega_1 + \omega_1 u - u\omega_2 + \frac{1}{2}\omega_2 & 0 & -1 + \omega_1 + 2\omega_1 u - 2u\omega_2 \\ -1 + \frac{1}{2}\omega_1 - \omega_1 u + u\omega_2 + \frac{1}{2}\omega_2 & -1 + \omega_1 - 2\omega_1 u + 2u\omega_2 & 0 \end{pmatrix}.$$

4.2 EFDE for μ_1

$$\mu_{1,x}^{t+\delta_t} = \sum_{\ell=0}^{2} \sum_{i=-\ell-1}^{\ell+1} \alpha_{x+i\delta_l}^{[\mu_1],t-\ell\delta_t} \mu_{x+i\delta_l}^{t-\ell\delta_t},$$

where the non-zero coefficients are given by:

$\alpha_{x-\delta_l}^{[\mu_1],t} = 1 + u^2\omega_1 - \frac{1}{2}\omega_2 + \frac{1}{2}\omega_2 {c_s}^2 + \omega_2 u - \frac{1}{2}\omega_1 - \frac{1}{2}\omega_2 u^2 - \frac{1}{2}u\omega_1.$

$\alpha_{x}^{[\mu_1],t} = 1 - 2u^2\omega_1 - \omega_2 {c_s}^2 + \omega_2 u^2.$

$\alpha_{x+\delta_l}^{[\mu_1],t} = 1 + u^2\omega_1 - \frac{1}{2}\omega_2 + \frac{1}{2}\omega_2 {c_s}^2 - \omega_2 u - \frac{1}{2}\omega_1 - \frac{1}{2}\omega_2 u^2 + \frac{1}{2}u\omega_1.$

$\alpha_{x-\delta_l}^{[\mu_1],t-\delta_t} = -1 - \frac{1}{2}\omega_2 {c_s}^2\omega_1 + u^2\omega_1 + \frac{1}{2}\omega_2 - \frac{1}{2}\omega_2 u^2\omega_1 + \frac{1}{2}\omega_2 {c_s}^2 - \omega_2 u + \frac{1}{2}\omega_2 u\omega_1 + \frac{1}{2}\omega_1 - \frac{1}{2}\omega_2 u^2 + \frac{1}{2}u\omega_1,$

$\alpha_{x}^{[\mu_1],t-\delta_t} = -1 + \omega_2 {c_s}^2\omega_1 - 2u^2\omega_1 + \omega_2 + \omega_2 u^2\omega_1 - \omega_2 {c_s}^2 - \omega_2\omega_1 + \omega_1 + \omega_2 u^2,$

$\alpha_{x+\delta_l}^{[\mu_1],t-\delta_t} = -1 - \frac{1}{2}\omega_2 {c_s}^2\omega_1 + u^2\omega_1 + \frac{1}{2}\omega_2 - \frac{1}{2}\omega_2 u^2\omega_1 + \frac{1}{2}\omega_2 {c_s}^2 + \omega_2 u - \frac{1}{2}\omega_2 u\omega_1 + \frac{1}{2}\omega_1 - \frac{1}{2}\omega_2 u^2 - \frac{1}{2}u\omega_1,$

$\alpha_{x}^{[\mu_1],t-2\delta_t} = 1 - \omega_2 + \omega_2\omega_1 - \omega_1,$

4.3 EFDE for μ_2

$$\mu_{2,x}^{t+\delta_t} = \sum_{\ell=0}^{2} \sum_{i=-\ell-1}^{\ell+1} \alpha_{x+i\delta_l}^{[\mu_1],t-\ell\delta_t} \mu_{x+i\delta_l}^{t-\ell\delta_t} + \sum_{\ell=0}^{2} \sum_{i=-\ell-1}^{\ell+1} \alpha_{x+i\delta_l}^{[\mu_2],t-\ell\delta_t} \mu_{2,x+i\delta_l}^{t-\ell\delta_t},$$

where the non-zero coefficients are given by:

$\alpha_{x-\delta_l}^{[\mu_1],t} = 1 + u^2\omega_1 - \frac{1}{2}\omega_2 + \frac{1}{2}\omega_2 {c_s}^2 + \omega_2 u - \frac{1}{2}\omega_1 - \frac{1}{2}\omega_2 u^2 - \frac{1}{2}u\omega_1.$

$\alpha_{x+\delta_l}^{[\mu_1],t} = -1 - u^2\omega_1 + \frac{1}{2}\omega_2 - \frac{1}{2}\omega_2 {c_s}^2 + \omega_2 u + \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2 u^2 - \frac{1}{2}u\omega_1.$

$\alpha_{x-\delta_l}^{[\mu_1],t-\delta_t} = -1 - \frac{1}{2}\omega_2 {c_s}^2\omega_1 + 3\omega_2 u^3\omega_1 + \frac{1}{2}\omega_2^2 u^2 + 2u^2\omega_1 + \frac{1}{2}\omega_2 + \omega_2^2 {c_s}^2 u - 2u^3\omega_1^2 - \omega_2^2 u^3 - \frac{1}{2}\omega_2 u^2\omega_1 + \omega_2 {c_s}^2 - u^2\omega_1^2 - \omega_2 {c_s}^2 u\omega_1 - \omega_2 u - \frac{1}{2}\omega_2^2 {c_s}^2 + \frac{1}{2}\omega_1 - \omega_2 u^2 + u\omega_1,$

$\alpha_{x-\delta_l}^{[\mu_2],t-\delta_t} = 1 + 2\omega_2^2 u^2 - \frac{3}{2}\omega_2 - 4\omega_2 u^2\omega_1 + 2u^2\omega_1^2 - 2\omega_2^2 u + \frac{1}{2}\omega_2\omega_1 + u\omega_1^2 + 3\omega_2 u + \omega_2 u\omega_1 - \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2^2 - 3u\omega_1,$

$\alpha_{x}^{[\mu_1],t-\delta_t} = -6\omega_2 u^3\omega_1 - 2\omega_2^2 {c_s}^2 u + 4u^3\omega_1^2 + 2\omega_2^2 u^3 + 2\omega_2^2 u + 2\omega_2 {c_s}^2 u\omega_1 - u\omega_1^2 - 2\omega_2 u - 2\omega_2 u\omega_1 + 3u\omega_1,$

$\alpha_{x}^{[\mu_2],t-\delta_t} = 1 - 4\omega_2^2 u^2 + \omega_1^2 + 8\omega_2 u^2\omega_1 - 4u^2\omega_1^2 - 2\omega_1,$

$\alpha_{x+\delta_l}^{[\mu_1],t-\delta_t} = 1 + \frac{1}{2}\omega_2 {c_s}^2\omega_1 + 3\omega_2 u^3\omega_1 - \frac{1}{2}\omega_2^2 u^2 - 2u^2\omega_1 - \frac{1}{2}\omega_2 + \omega_2^2 {c_s}^2 u - 2u^3\omega_1^2 - \omega_2^2 u^3 + \frac{1}{2}\omega_2 u^2\omega_1 - \omega_2 {c_s}^2 + u^2\omega_1^2 - \omega_2 {c_s}^2 u\omega_1 - \omega_2 u + \frac{1}{2}\omega_2^2 {c_s}^2 - \frac{1}{2}\omega_1 + \omega_2 u^2 + u\omega_1,$

$\alpha_{x+\delta_l}^{[\mu_2],t-\delta_t} = 1 + 2\omega_2^2 u^2 - \frac{3}{2}\omega_2 - 4\omega_2 u^2\omega_1 + 2u^2\omega_1^2 + 2\omega_2^2 u + \frac{1}{2}\omega_2\omega_1 - u\omega_1^2 - 3\omega_2 u - \omega_2 u\omega_1 - \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2^2 + 3u\omega_1,$

$$\alpha_{x}^{[\mu_1],t-2\delta_t} = -2\omega_2^2 u - \omega_2^2 u\omega_1 + u\omega_1^2 + 2\omega_2 u + 5\omega_2 u\omega_1 - \omega_2 u\omega_1^2 - 4u\omega_1,$$
$$\alpha_{x}^{[\mu_2],t-2\delta_t} = -2 + 3\omega_2 + \omega_2\omega_1^2 - \omega_1^2 - 4\omega_2\omega_1 + \omega_2^2\omega_1 + 3\omega_1 - \omega_2^2,$$

4.4 EFDE for μ_3

$$\mu_{3,x}^{t+\delta_t} = \sum_{\ell=0}^{2}\sum_{i=-\ell-1}^{\ell+1} \alpha_{x+i\delta_l}^{[\mu_1],t-\ell\delta_t}\mu_{x+i\delta_l}^{t-\ell\delta_t} + \sum_{\ell=0}^{2}\sum_{i=-\ell-1}^{\ell+1} \alpha_{x+i\delta_l}^{[\mu_3],t-\ell\delta_t}\mu_{3,x+i\delta_l}^{t-\ell\delta_t},$$

where the non-zero coefficients are given by:

$$\alpha_{x-\delta_l}^{[\mu_1],t} = 1 + u^2\omega_1 - \tfrac{1}{2}\omega_2 + \tfrac{1}{2}\omega_2 {c_s}^2 + \omega_2 u - \tfrac{1}{2}\omega_1 - \tfrac{1}{2}\omega_2 u^2 - \tfrac{1}{2}u\omega_1.$$
$$\alpha_{x+\delta_l}^{[\mu_1],t} = 1 + u^2\omega_1 - \tfrac{1}{2}\omega_2 + \tfrac{1}{2}\omega_2 {c_s}^2 - \omega_2 u - \tfrac{1}{2}\omega_1 - \tfrac{1}{2}\omega_2 u^2 + \tfrac{1}{2}u\omega_1.$$
$$\alpha_{x-\delta_l}^{[\mu_1],t-\delta_t} = -1 - \tfrac{1}{2}\omega_2 {c_s}^2\omega_1 + 3\omega_2 u^3\omega_1 + \tfrac{1}{2}\omega_2^2 u^2 + 2u^2\omega_1 + \tfrac{1}{2}\omega_2 + \omega_2^2 {c_s}^2 u - 2u^3\omega_1^2 - \omega_2^2 u^3 - \tfrac{1}{2}\omega_2 u^2\omega_1 + \omega_2 {c_s}^2 - u^2\omega_1^2 - \omega_2 {c_s}^2 u\omega_1 - \omega_2 u - \tfrac{1}{2}\omega_2^2 {c_s}^2 + \tfrac{1}{2}\omega_1 - \omega_2 u^2 + u\omega_1,$$
$$\alpha_{x-\delta_l}^{[\mu_3],t-\delta_t} = 1 + 2\omega_2^2 u^2 - \tfrac{3}{2}\omega_2 - 4\omega_2 u^2\omega_1 + 2u^2\omega_1^2 - 2\omega_2^2 u + \tfrac{1}{2}\omega_2\omega_1 + u\omega_1^2 + 3\omega_2 u + \omega_2 u\omega_1 - \tfrac{1}{2}\omega_1 + \tfrac{1}{2}\omega_2^2 - 3u\omega_1,$$
$$\alpha_{x}^{[\mu_1],t-\delta_t} = -2 + \omega_2 {c_s}^2\omega_1 + 4\omega_2^2 u^2 - 2u^2\omega_1 + \omega_2 - \omega_1^2 - 7\omega_2 u^2\omega_1 - \omega_2 {c_s}^2 + 4u^2\omega_1^2 - \omega_2\omega_1 + 3\omega_1 + \omega_2 u^2,$$
$$\alpha_{x}^{[\mu_3],t-\delta_t} = 1 - 4\omega_2^2 u^2 + \omega_1^2 + 8\omega_2 u^2\omega_1 - 4u^2\omega_1^2 - 2\omega_1,$$
$$\alpha_{x+\delta_l}^{[\mu_1],t-\delta_t} = -1 - \tfrac{1}{2}\omega_2 {c_s}^2\omega_1 - 3\omega_2 u^3\omega_1 + \tfrac{1}{2}\omega_2^2 u^2 + 2u^2\omega_1 + \tfrac{1}{2}\omega_2 - \omega_2^2 {c_s}^2 u + 2u^3\omega_1^2 + \omega_2^2 u^3 - \tfrac{1}{2}\omega_2 u^2\omega_1 + \omega_2 {c_s}^2 - u^2\omega_1^2 + \omega_2 {c_s}^2 u\omega_1 + \omega_2 u - \tfrac{1}{2}\omega_2^2 {c_s}^2 + \tfrac{1}{2}\omega_1 - \omega_2 u^2 - u\omega_1,$$
$$\alpha_{x+\delta_l}^{[\mu_3],t-\delta_t} = 1 + 2\omega_2^2 u^2 - \tfrac{3}{2}\omega_2 - 4\omega_2 u^2\omega_1 + 2u^2\omega_1^2 + 2\omega_2^2 u + \tfrac{1}{2}\omega_2\omega_1 - u\omega_1^2 - 3\omega_2 u - \omega_2 u\omega_1 - \tfrac{1}{2}\omega_1 + \tfrac{1}{2}\omega_2^2 + 3u\omega_1,$$
$$\alpha_{x}^{[\mu_1],t-2\delta_t} = 2 + 3\omega_2 {c_s}^2\omega_1 - 5\omega_2^2 u^2 - 4u^2\omega_1 - \omega_2 - \omega_2 u^2\omega_1^2 + \omega_1^2 + 11\omega_2 u^2\omega_1 - 2\omega_2 {c_s}^2 - 2u^2\omega_1^2 + \omega_2\omega_1 - \omega_2 {c_s}^2\omega_1^2 - \omega_2^2 u^2\omega_1 + \omega_2^2 {c_s}^2 - 3\omega_1 - \omega_2^2 {c_s}^2\omega_1 + 2\omega_2 u^2,$$
$$\alpha_{x}^{[\mu_3],t-2\delta_t} = -2 + 3\omega_2 + \omega_2\omega_1^2 - \omega_1^2 - 4\omega_2\omega_1 + \omega_2^2\omega_1 + 3\omega_1 - \omega_2^2,$$