EPDE for D1Q3 with constant velocities,
supplementary material for
Equivalent Finite Difference Equations and Equivalent Partial Differential Equations for the Lattice Boltzmann Method

Radek Fučík[†] and Robert Straka[‡,†]

[†]Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Trojanova 13, 120 00 Prague, Czech Republic
[‡]AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Krakow, Poland

Contents

1 Global definitions

In $\mathbb{R}$, the position and velocity vectors are given by $\boldsymbol{x} = (x)$ and $\boldsymbol{u} = (u)$, respectively.
Discrete velocity vectors:

$$\{\boldsymbol{c}_i\}_{i=1}^{3} = \left((0)\,, (1)\,, (-1)\right).$$

Equilibrium DF vector $\boldsymbol{f}^{eq}$:

$$\boldsymbol{f}^{eq} = \begin{pmatrix} 1 - c_s^2 - u^2 \\ \frac{1}{2}u + \frac{1}{2}c_s^2 + \frac{1}{2}u^2 \\ -\frac{1}{2}u + \frac{1}{2}c_s^2 + \frac{1}{2}u^2 \end{pmatrix}.$$

Lattice speed of sound: $c_s = \frac{1}{\sqrt{3}}$.
Moments $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)^T$ are given by

$$\boldsymbol{\mu} = \tilde{\mathbf{M}}\boldsymbol{f},$$

where $\boldsymbol{f} = (f_1, f_2, f_3)^T$ and

$$\tilde{\mathbf{M}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

2 SRT

2.1 Definitions

Matrix $\mathbf{A} = \mathbf{S}$:

$$\mathbf{A} = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

where

$$\mathbf{S} = \mathrm{diag}(\omega, \omega, \omega).$$

Matrix $\mathbf{B}$:

$$\mathbf{B} = \begin{pmatrix} 0 & -1+\omega & -1+\omega \\ -1+\omega & 0 & -1+\omega \\ -1+\omega & -1+\omega & 0 \end{pmatrix}.$$

2.2 EPDE for μ_1

$$\boxed{\gamma_{[t]}^{[\mu_1]}\delta_t \frac{\partial \mu_1}{\partial t} + \gamma_{[x]}^{[\mu_1]}\delta_l \frac{\partial \mu_1}{\partial x} + \gamma_{[t^2]}^{[\mu_1]}{\delta_t}^2 \frac{\partial^2 \mu_1}{\partial t^2} + \gamma_{[tx]}^{[\mu_1]}\delta_t\delta_l \frac{\partial^2 \mu_1}{\partial t \partial x} + \gamma_{[x^2]}^{[\mu_1]}{\delta_l}^2 \frac{\partial^2 \mu_1}{\partial x^2} = 0,}$$

where
$\gamma_{[t]}^{[\mu_1]} = -\omega^2,$

$\gamma_{[x]}^{[\mu_1]} = -u\omega^2,$

$\gamma_{[t^2]}^{[\mu_1]} = -2\omega + \frac{3}{2}\omega^2,$

$\gamma_{[tx]}^{[\mu_1]} = u\omega^2 - u\omega.$

$\gamma_{[x^2]}^{[\mu_1]} = {c_s}^2\omega + u^2\omega - \frac{1}{2}u^2\omega^2 - \frac{1}{2}{c_s}^2\omega^2,$

2.3 EPDE for μ_2

$$\gamma_{[1]}^{[\mu_1]}\mu_1 + \gamma_{[1]}^{[\mu_2]}\mu_2 + \gamma_{[t]}^{[\mu_1]}\delta_t\frac{\partial\mu_1}{\partial t} + \gamma_{[t]}^{[\mu_2]}\delta_t\frac{\partial\mu_2}{\partial t} + \gamma_{[x]}^{[\mu_1]}\delta_l\frac{\partial\mu_1}{\partial x} + \gamma_{[t^2]}^{[\mu_1]}{\delta_t}^2\frac{\partial^2\mu_1}{\partial t^2}$$
$$+ \gamma_{[t^2]}^{[\mu_2]}{\delta_t}^2\frac{\partial^2\mu_2}{\partial t^2} + \gamma_{[tx]}^{[\mu_1]}\delta_t\delta_l\frac{\partial^2\mu_1}{\partial t\partial x} + \gamma_{[x^2]}^{[\mu_1]}{\delta_l}^2\frac{\partial^2\mu_1}{\partial x^2} + \gamma_{[x^2]}^{[\mu_2]}{\delta_l}^2\frac{\partial^2\mu_2}{\partial x^2} = 0,$$

where
$\gamma_{[1]}^{[\mu_1]} = -2u\omega^3 + 3u\omega^2,$

$\gamma_{[1]}^{[\mu_2]} = -3\omega^2 + 2\omega^3,$

$\gamma_{[t]}^{[\mu_1]} = 4u\omega^3 - 7u\omega^2 + 3u\omega,$

$\gamma_{[t]}^{[\mu_2]} = -6\omega + 9\omega^2 - 4\omega^3,$

$\gamma_{[x]}^{[\mu_1]} = -3{c_s}^2\omega - 3u^2\omega + 2u^2\omega^2 + 2{c_s}^2\omega^2,$

$\gamma_{[t^2]}^{[\mu_1]} = -4u\omega^3 + \frac{15}{2}u\omega^2 - \frac{7}{2}u\omega,$

$\gamma_{[t^2]}^{[\mu_2]} = -3 + 9\omega - \frac{21}{2}\omega^2 + 4\omega^3,$

$\gamma_{[tx]}^{[\mu_1]} = -2 + 2\omega + 2{c_s}^2\omega + 2u^2\omega - 2u^2\omega^2 - 2{c_s}^2\omega^2,$

$\gamma_{[x^2]}^{[\mu_1]} = \frac{1}{2}u\omega.$

$\gamma_{[x^2]}^{[\mu_2]} = 1 - 2\omega + \omega^2,$

2.4 EPDE for μ_3

$$\gamma_{[1]}^{[\mu_1]}\mu_1 + \gamma_{[1]}^{[\mu_3]}\mu_3 + \gamma_{[t]}^{[\mu_1]}\delta_t\frac{\partial\mu_1}{\partial t} + \gamma_{[t]}^{[\mu_3]}\delta_t\frac{\partial\mu_3}{\partial t} + \gamma_{[x]}^{[\mu_1]}\delta_l\frac{\partial\mu_1}{\partial x}$$
$$+ \gamma_{[t^2]}^{[\mu_1]}{\delta_t}^2\frac{\partial^2\mu_1}{\partial t^2} + \gamma_{[t^2]}^{[\mu_3]}{\delta_t}^2\frac{\partial^2\mu_3}{\partial t^2} + \gamma_{[x^2]}^{[\mu_1]}{\delta_l}^2\frac{\partial^2\mu_1}{\partial x^2} + \gamma_{[x^2]}^{[\mu_3]}{\delta_l}^2\frac{\partial^2\mu_3}{\partial x^2} = 0,$$

where
$\gamma_{[1]}^{[\mu_1]} = -2{c_s}^2\omega^3 + 3u^2\omega^2 + 3{c_s}^2\omega^2 - 2u^2\omega^3,$

$\gamma_{[1]}^{[\mu_3]} = -3\omega^2 + 2\omega^3,$

$\gamma_{[t]}^{[\mu_1]} = 2\omega + 3{c_s}^2\omega + 3u^2\omega - 2\omega^2 + 4{c_s}^2\omega^3 - 7u^2\omega^2 - 7{c_s}^2\omega^2 + 4u^2\omega^3,$

$\gamma_{[t]}^{[\mu_3]} = -6\omega + 9\omega^2 - 4\omega^3,$

$\gamma_{[x]}^{[\mu_1]} = -u\omega,$

$\gamma_{[t^2]}^{[\mu_1]} = 2 - 5\omega - \frac{7}{2}{c_s}^2\omega - \frac{7}{2}u^2\omega + 3\omega^2 - 4{c_s}^2\omega^3 + \frac{15}{2}u^2\omega^2 + \frac{15}{2}{c_s}^2\omega^2 - 4u^2\omega^3,$

$\gamma_{[t^2]}^{[\mu_3]} = -3 + 9\omega - \frac{21}{2}\omega^2 + 4\omega^3,$

$\gamma_{[x^2]}^{[\mu_1]} = \frac{3}{2}{c_s}^2\omega + \frac{3}{2}u^2\omega - u^2\omega^2 - {c_s}^2\omega^2.$

$\gamma_{[x^2]}^{[\mu_3]} = 1 - 2\omega + \omega^2,$

3 MRT

3.1 Definitions

Matrix $\mathbf{A} = \mathbf{M}^{-1}\mathbf{S}\mathbf{M}$:

$$\mathbf{A} = \begin{pmatrix} \omega_0 & \omega_0 - \omega_2 & \omega_0 - \omega_2 \\ 0 & \frac{1}{2}\omega_2 + \frac{1}{2}\omega_1 & \frac{1}{2}\omega_2 - \frac{1}{2}\omega_1 \\ 0 & \frac{1}{2}\omega_2 - \frac{1}{2}\omega_1 & \frac{1}{2}\omega_2 + \frac{1}{2}\omega_1 \end{pmatrix},$$

where

$$\mathbf{S} = \mathrm{diag}(\omega_0, \omega_1, \omega_2)$$

and

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Matrix $\mathbf{B}$:

$$\mathbf{B} = \begin{pmatrix} 0 & -1 + \omega_2 & -1 + \omega_2 \\ -1 + \frac{1}{2}\omega_2 + \frac{1}{2}\omega_1 & 0 & -1 + \omega_1 \\ -1 + \frac{1}{2}\omega_2 + \frac{1}{2}\omega_1 & -1 + \omega_1 & 0 \end{pmatrix}.$$

3.2 EPDE for μ_1

$$\boxed{\gamma_{[t]}^{[\mu_1]}\delta_t\frac{\partial \mu_1}{\partial t} + \gamma_{[x]}^{[\mu_1]}\delta_l\frac{\partial \mu_1}{\partial x} + \gamma_{[t^2]}^{[\mu_1]}{\delta_t}^2\frac{\partial^2 \mu_1}{\partial t^2} + \gamma_{[tx]}^{[\mu_1]}\delta_t\delta_l\frac{\partial^2 \mu_1}{\partial t \partial x} + \gamma_{[x^2]}^{[\mu_1]}{\delta_l}^2\frac{\partial^2 \mu_1}{\partial x^2} = 0,}$$

where

$\gamma_{[t]}^{[\mu_1]} = -\omega_2\omega_1,$

$\gamma_{[x]}^{[\mu_1]} = -\omega_2 u\omega_1,$

$\gamma_{[t^2]}^{[\mu_1]} = -\omega_2 + \frac{3}{2}\omega_2\omega_1 - \omega_1,$

$\gamma_{[tx]}^{[\mu_1]} = -u\omega_1 + \omega_2 u\omega_1.$

$\gamma_{[x^2]}^{[\mu_1]} = \omega_2{c_s}^2 + \omega_2 u^2 - \frac{1}{2}\omega_2 u^2\omega_1 - \frac{1}{2}\omega_2{c_s}^2\omega_1,$

3.3 EPDE for μ_2

$$\gamma_{[1]}^{[\mu_1]}\mu_1 + \gamma_{[1]}^{[\mu_2]}\mu_2 + \gamma_{[t]}^{[\mu_1]}\delta_t\frac{\partial\mu_1}{\partial t} + \gamma_{[t]}^{[\mu_2]}\delta_t\frac{\partial\mu_2}{\partial t} + \gamma_{[x]}^{[\mu_1]}\delta_l\frac{\partial\mu_1}{\partial x} + \gamma_{[t^2]}^{[\mu_1]}{\delta_t}^2\frac{\partial^2\mu_1}{\partial t^2}$$
$$+ \gamma_{[t^2]}^{[\mu_2]}{\delta_t}^2\frac{\partial^2\mu_2}{\partial t^2} + \gamma_{[tx]}^{[\mu_1]}\delta_t\delta_l\frac{\partial^2\mu_1}{\partial t\partial x} + \gamma_{[x^2]}^{[\mu_1]}{\delta_l}^2\frac{\partial^2\mu_1}{\partial x^2} + \gamma_{[x^2]}^{[\mu_2]}{\delta_l}^2\frac{\partial^2\mu_2}{\partial x^2} = 0,$$

where

$\gamma_{[1]}^{[\mu_1]} = -\omega_2^2u\omega_1 + 3\omega_2u\omega_1 - \omega_2u\omega_1^2,$

$\gamma_{[1]}^{[\mu_2]} = \omega_2\omega_1^2 - 3\omega_2\omega_1 + \omega_2^2\omega_1,$

$\gamma_{[t]}^{[\mu_1]} = -u\omega_1^2 + 3u\omega_1 + 2\omega_2^2u\omega_1 - 6\omega_2u\omega_1 + 2\omega_2u\omega_1^2,$

$\gamma_{[t]}^{[\mu_2]} = -2\omega_2\omega_1^2 + \omega_1^2 - 3\omega_2 + \omega_2^2 + 7\omega_2\omega_1 - 2\omega_2^2\omega_1 - 3\omega_1,$

$\gamma_{[x]}^{[\mu_1]} = -3\omega_2{c_s}^2 - 3\omega_2u^2 + \omega_2u^2\omega_1 + \omega_2{c_s}^2\omega_1 + \omega_2^2u^2 + \omega_2^2{c_s}^2,$

$\gamma_{[t^2]}^{[\mu_1]} = \frac{3}{2}u\omega_1^2 - \frac{7}{2}u\omega_1 - 2\omega_2^2u\omega_1 + 6\omega_2u\omega_1 - 2\omega_2u\omega_1^2,$

$\gamma_{[t^2]}^{[\mu_2]} = -3 + 2\omega_2\omega_1^2 - \frac{3}{2}\omega_1^2 + \frac{9}{2}\omega_2 - \frac{3}{2}\omega_2^2 - \frac{15}{2}\omega_2\omega_1 + 2\omega_2^2\omega_1 + \frac{9}{2}\omega_1,$

$\gamma_{[tx]}^{[\mu_1]} = -2 + 2\omega_2{c_s}^2 + 2\omega_2u^2 + \omega_2 - \omega_2u^2\omega_1 - \omega_2{c_s}^2\omega_1 - \omega_2^2u^2 - \omega_2^2{c_s}^2 + \omega_1,$

$\gamma_{[x^2]}^{[\mu_1]} = \frac{1}{2}u\omega_1.$

$\gamma_{[x^2]}^{[\mu_2]} = 1 - \frac{3}{2}\omega_2 + \frac{1}{2}\omega_2^2 + \frac{1}{2}\omega_2\omega_1 - \frac{1}{2}\omega_1,$

3.4 EPDE for μ_3

$$\gamma_{[1]}^{[\mu_1]}\mu_1 + \gamma_{[1]}^{[\mu_3]}\mu_3 + \gamma_{[t]}^{[\mu_1]}\delta_t\frac{\partial\mu_1}{\partial t} + \gamma_{[t]}^{[\mu_3]}\delta_t\frac{\partial\mu_3}{\partial t} + \gamma_{[x]}^{[\mu_1]}\delta_l\frac{\partial\mu_1}{\partial x}$$
$$+ \gamma_{[t^2]}^{[\mu_1]}{\delta_t}^2\frac{\partial^2\mu_1}{\partial t^2} + \gamma_{[t^2]}^{[\mu_3]}{\delta_t}^2\frac{\partial^2\mu_3}{\partial t^2} + \gamma_{[x^2]}^{[\mu_1]}{\delta_l}^2\frac{\partial^2\mu_1}{\partial x^2} + \gamma_{[x^2]}^{[\mu_3]}{\delta_l}^2\frac{\partial^2\mu_3}{\partial x^2} = 0,$$

where

$\gamma_{[1]}^{[\mu_1]} = -\omega_2u^2\omega_1^2 - \omega_2^2{c_s}^2\omega_1 + 3\omega_2u^2\omega_1 - \omega_2^2u^2\omega_1 + 3\omega_2{c_s}^2\omega_1 - \omega_2{c_s}^2\omega_1^2,$

$\gamma_{[1]}^{[\mu_3]} = \omega_2\omega_1^2 - 3\omega_2\omega_1 + \omega_2^2\omega_1,$

$\gamma_{[t]}^{[\mu_1]} = 3\omega_2{c_s}^2 - \omega_1^2 + 2\omega_2u^2\omega_1^2 + 3\omega_2u^2 + 2\omega_2^2{c_s}^2\omega_1 - 6\omega_2u^2\omega_1 - \omega_2\omega_1 + 2\omega_2^2u^2\omega_1 - 6\omega_2{c_s}^2\omega_1 - \omega_2^2u^2 - \omega_2^2{c_s}^2 + 2\omega_1 + 2\omega_2{c_s}^2\omega_1^2,$

$$\gamma_{[t]}^{[\mu_3]} = -2\omega_2\omega_1^2 + \omega_1^2 - 3\omega_2 + \omega_2^2 + 7\omega_2\omega_1 - 2\omega_2^2\omega_1 - 3\omega_1,$$
$$\gamma_{[x]}^{[\mu_1]} = -u\omega_1,$$
$$\gamma_{[t^2]}^{[\mu_1]} = 2 - \tfrac{7}{2}\omega_2 {c_s}^2 + \tfrac{3}{2}\omega_1^2 - 2\omega_2 u^2\omega_1^2 - \tfrac{7}{2}\omega_2 u^2 - \omega_2 - 2\omega_2^2 {c_s}^2\omega_1 + 6\omega_2 u^2\omega_1 + \tfrac{3}{2}\omega_2\omega_1 - 2\omega_2^2 u^2\omega_1 + 6\omega_2 {c_s}^2\omega_1 + \tfrac{3}{2}\omega_2^2 u^2 + \tfrac{3}{2}\omega_2^2 {c_s}^2 - 4\omega_1 - 2\omega_2 {c_s}^2\omega_1^2,$$
$$\gamma_{[t^2]}^{[\mu_3]} = -3 + 2\omega_2\omega_1^2 - \tfrac{3}{2}\omega_1^2 + \tfrac{9}{2}\omega_2 - \tfrac{3}{2}\omega_2^2 - \tfrac{15}{2}\omega_2\omega_1 + 2\omega_2^2\omega_1 + \tfrac{9}{2}\omega_1,$$
$$\gamma_{[x^2]}^{[\mu_1]} = \tfrac{3}{2}\omega_2 {c_s}^2 + \tfrac{3}{2}\omega_2 u^2 - \tfrac{1}{2}\omega_2 u^2\omega_1 - \tfrac{1}{2}\omega_2 {c_s}^2\omega_1 - \tfrac{1}{2}\omega_2^2 u^2 - \tfrac{1}{2}\omega_2^2 {c_s}^2.$$
$$\gamma_{[x^2]}^{[\mu_3]} = 1 - \tfrac{3}{2}\omega_2 + \tfrac{1}{2}\omega_2^2 + \tfrac{1}{2}\omega_2\omega_1 - \tfrac{1}{2}\omega_1,$$

4 CLBM

4.1 Definitions

Matrix $\mathbf{A} = \mathbf{K}^{-1}\mathbf{S}\mathbf{K}$:

$$\begin{aligned}
\mathbf{A}_{1,1} &= 2\omega_1 u^2 - u^2\omega_2 + \omega_0 - \omega_0 u^2,\\
\mathbf{A}_{1,2} &= (2\omega_1 u - \omega_0 - u(\omega_0 + \omega_2) + \omega_2)(-1 + u),\\
\mathbf{A}_{1,3} &= (2\omega_1 u + \omega_0 - u(\omega_0 + \omega_2) - \omega_2)(1 + u),\\
\mathbf{A}_{2,1} &= -\frac{1}{2}(\omega_1(1 + 2u) - \omega_0 - u(\omega_0 + \omega_2))u,\\
\mathbf{A}_{2,2} &= -\omega_1 u^2 + \frac{1}{2}\omega_1 + \frac{1}{2}\omega_1 u + \frac{1}{2}u^2\omega_2 - u\omega_2 + \frac{1}{2}\omega_0 u^2 + \frac{1}{2}\omega_2 + \frac{1}{2}\omega_0 u,\\
\mathbf{A}_{2,3} &= -\frac{1}{2}(\omega_1(1 + 2u) - u(\omega_0 + \omega_2) - \omega_2)(1 + u),\\
\mathbf{A}_{3,1} &= -\frac{1}{2}(\omega_0 + \omega_1(-1 + 2u) - u(\omega_0 + \omega_2))u,\\
\mathbf{A}_{3,2} &= -\frac{1}{2}(-1 + u)(\omega_1(-1 + 2u) - u(\omega_0 + \omega_2) + \omega_2),\\
\mathbf{A}_{3,3} &= -\omega_1 u^2 + \frac{1}{2}\omega_1 - \frac{1}{2}\omega_1 u + \frac{1}{2}u^2\omega_2 + u\omega_2 + \frac{1}{2}\omega_0 u^2 + \frac{1}{2}\omega_2 - \frac{1}{2}\omega_0 u.
\end{aligned}$$

where
$$\mathbf{S} = \mathrm{diag}(\omega_0, \omega_1, \omega_2)$$
and
$$\mathbf{K} = \begin{pmatrix} 1 & 1 & 1 \\ -u & 1-u & -u-1 \\ u^2 & (1-u)^2 & (u+1)^2 \end{pmatrix}.$$

Matrix $\mathbf{B}$:
$$\mathbf{B} = \begin{pmatrix} 0 & -1 + 2\omega_1 u - 2u\omega_2 + \omega_2 & -1 - 2\omega_1 u + 2u\omega_2 + \omega_2 \\ -1 + \frac{1}{2}\omega_1 + \omega_1 u - u\omega_2 + \frac{1}{2}\omega_2 & 0 & -1 + \omega_1 + 2\omega_1 u - 2u\omega_2 \\ -1 + \frac{1}{2}\omega_1 - \omega_1 u + u\omega_2 + \frac{1}{2}\omega_2 & -1 + \omega_1 - 2\omega_1 u + 2u\omega_2 & 0 \end{pmatrix}.$$

4.2 EPDE for μ_1

$$\gamma_{[t]}^{[\mu_1]}\delta_t\frac{\partial\mu_1}{\partial t}+\gamma_{[x]}^{[\mu_1]}\delta_l\frac{\partial\mu_1}{\partial x}+\gamma_{[t^2]}^{[\mu_1]}{\delta_t}^2\frac{\partial^2\mu_1}{\partial t^2}+\gamma_{[tx]}^{[\mu_1]}\delta_t\delta_l\frac{\partial^2\mu_1}{\partial t\partial x}+\gamma_{[x^2]}^{[\mu_1]}{\delta_l}^2\frac{\partial^2\mu_1}{\partial x^2}=0,$$

where
$\gamma_{[t]}^{[\mu_1]}=-\omega_2\omega_1,$

$\gamma_{[x]}^{[\mu_1]}=-\omega_2u\omega_1,$

$\gamma_{[t^2]}^{[\mu_1]}=-\omega_2+\frac{3}{2}\omega_2\omega_1-\omega_1,$

$\gamma_{[tx]}^{[\mu_1]}=-2\omega_2u+\omega_2u\omega_1+u\omega_1.$

$\gamma_{[x^2]}^{[\mu_1]}=-\frac{1}{2}\omega_2{c_s}^2\omega_1+2u^2\omega_1-\frac{1}{2}\omega_2u^2\omega_1+\omega_2{c_s}^2-\omega_2u^2,$

4.3 EPDE for μ_2

$$\gamma_{[1]}^{[\mu_1]}\mu_1+\gamma_{[1]}^{[\mu_2]}\mu_2+\gamma_{[t]}^{[\mu_1]}\delta_t\frac{\partial\mu_1}{\partial t}+\gamma_{[t]}^{[\mu_2]}\delta_t\frac{\partial\mu_2}{\partial t}+\gamma_{[x]}^{[\mu_1]}\delta_l\frac{\partial\mu_1}{\partial x}+\gamma_{[x]}^{[\mu_2]}\delta_l\frac{\partial\mu_2}{\partial x}+\gamma_{[t^2]}^{[\mu_1]}{\delta_t}^2\frac{\partial^2\mu_1}{\partial t^2}$$
$$+\gamma_{[t^2]}^{[\mu_2]}{\delta_t}^2\frac{\partial^2\mu_2}{\partial t^2}+\gamma_{[tx]}^{[\mu_1]}\delta_t\delta_l\frac{\partial^2\mu_1}{\partial t\partial x}+\gamma_{[tx]}^{[\mu_2]}\delta_t\delta_l\frac{\partial^2\mu_2}{\partial t\partial x}+\gamma_{[x^2]}^{[\mu_1]}{\delta_l}^2\frac{\partial^2\mu_1}{\partial x^2}+\gamma_{[x^2]}^{[\mu_2]}{\delta_l}^2\frac{\partial^2\mu_2}{\partial x^2}=0,$$

where
$\gamma_{[1]}^{[\mu_1]}=-\omega_2^2u\omega_1+3\omega_2u\omega_1-\omega_2u\omega_1^2,$

$\gamma_{[1]}^{[\mu_2]}=\omega_2\omega_1^2-3\omega_2\omega_1+\omega_2^2\omega_1,$

$\gamma_{[t]}^{[\mu_1]}=2\omega_2^2u+2\omega_2^2u\omega_1-u\omega_1^2-8\omega_2u\omega_1+2\omega_2u\omega_1^2+3u\omega_1,$

$\gamma_{[t]}^{[\mu_2]}=-3\omega_2-2\omega_2\omega_1^2+\omega_1^2+7\omega_2\omega_1-2\omega_2^2\omega_1-3\omega_1+\omega_2^2,$

$\gamma_{[x]}^{[\mu_1]}=\omega_2{c_s}^2\omega_1-\omega_2^2u^2-6u^2\omega_1+\omega_2u^2\omega_1-3\omega_2{c_s}^2+2u^2\omega_1^2+\omega_2^2{c_s}^2+3\omega_2u^2,$

$\gamma_{[x]}^{[\mu_2]}=4\omega_2^2u-2u\omega_1^2-6\omega_2u-2\omega_2u\omega_1+6u\omega_1,$

$\gamma_{[t^2]}^{[\mu_1]}=-3\omega_2^2u-2\omega_2^2u\omega_1+\frac{3}{2}u\omega_1^2+2\omega_2u+9\omega_2u\omega_1-2\omega_2u\omega_1^2-\frac{11}{2}u\omega_1,$

$\gamma_{[t^2]}^{[\mu_2]}=-3+\frac{9}{2}\omega_2+2\omega_2\omega_1^2-\frac{3}{2}\omega_1^2-\frac{15}{2}\omega_2\omega_1+2\omega_2^2\omega_1+\frac{9}{2}\omega_1-\frac{3}{2}\omega_2^2,$

$\gamma_{[tx]}^{[\mu_1]}=-2-\omega_2{c_s}^2\omega_1+\omega_2^2u^2+4u^2\omega_1+\omega_2-\omega_2u^2\omega_1+2\omega_2{c_s}^2-2u^2\omega_1^2-\omega_2^2{c_s}^2+\omega_1-2\omega_2u^2,$

$\gamma_{[tx]}^{[\mu_2]}=-4\omega_2^2u+2u\omega_1^2+6\omega_2u+2\omega_2u\omega_1-6u\omega_1,$

$\gamma_{[x^2]}^{[\mu_1]} = 3\omega_2 u^3\omega_1 + \omega_2^2 {c_s}^2 u - 2u^3\omega_1^2 - \omega_2^2 u^3 - \omega_2 {c_s}^2 u\omega_1 + \frac{1}{2}u\omega_1.$

$\gamma_{[x^2]}^{[\mu_2]} = 1 + 2\omega_2^2 u^2 - \frac{3}{2}\omega_2 - 4\omega_2 u^2\omega_1 + 2u^2\omega_1^2 + \frac{1}{2}\omega_2\omega_1 - \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2^2,$

4.4 EPDE for μ_3

$$\gamma_{[1]}^{[\mu_1]}\mu_1 + \gamma_{[1]}^{[\mu_3]}\mu_3 + \gamma_{[t]}^{[\mu_1]}\delta_t\frac{\partial\mu_1}{\partial t} + \gamma_{[t]}^{[\mu_3]}\delta_t\frac{\partial\mu_3}{\partial t} + \gamma_{[x]}^{[\mu_1]}\delta_l\frac{\partial\mu_1}{\partial x} + \gamma_{[x]}^{[\mu_3]}\delta_l\frac{\partial\mu_3}{\partial x} + \gamma_{[t^2]}^{[\mu_1]}{\delta_t}^2\frac{\partial^2\mu_1}{\partial t^2}$$
$$+ \gamma_{[t^2]}^{[\mu_3]}{\delta_t}^2\frac{\partial^2\mu_3}{\partial t^2} + \gamma_{[tx]}^{[\mu_1]}\delta_t\delta_l\frac{\partial^2\mu_1}{\partial t\partial x} + \gamma_{[tx]}^{[\mu_3]}\delta_t\delta_l\frac{\partial^2\mu_3}{\partial t\partial x} + \gamma_{[x^2]}^{[\mu_1]}{\delta_l}^2\frac{\partial^2\mu_1}{\partial x^2} + \gamma_{[x^2]}^{[\mu_3]}{\delta_l}^2\frac{\partial^2\mu_3}{\partial x^2} = 0,$$

where

$\gamma_{[1]}^{[\mu_1]} = 3\omega_2 {c_s}^2\omega_1 - \omega_2 u^2\omega_1^2 + 3\omega_2 u^2\omega_1 - \omega_2 {c_s}^2\omega_1^2 - \omega_2^2 u^2\omega_1 - \omega_2^2 {c_s}^2\omega_1,$

$\gamma_{[1]}^{[\mu_3]} = \omega_2\omega_1^2 - 3\omega_2\omega_1 + \omega_2^2\omega_1,$

$\gamma_{[t]}^{[\mu_1]} = -6\omega_2 {c_s}^2\omega_1 + 5\omega_2^2 u^2 + 6u^2\omega_1 + 2\omega_2 u^2\omega_1^2 - \omega_1^2 - 14\omega_2 u^2\omega_1 + 3\omega_2 {c_s}^2 + 2u^2\omega_1^2 - \omega_2\omega_1 + 2\omega_2 {c_s}^2\omega_1^2 + 2\omega_2^2 u^2\omega_1 - \omega_2^2 {c_s}^2 + 2\omega_1 + 2\omega_2^2 {c_s}^2\omega_1 - 3\omega_2 u^2,$

$\gamma_{[t]}^{[\mu_3]} = -3\omega_2 - 2\omega_2\omega_1^2 + \omega_1^2 + 7\omega_2\omega_1 - 2\omega_2^2\omega_1 - 3\omega_1 + \omega_2^2,$

$\gamma_{[x]}^{[\mu_1]} = -6\omega_2 u^3\omega_1 - 2\omega_2^2 {c_s}^2 u + 4u^3\omega_1^2 + 2\omega_2^2 u^3 + 2\omega_2 {c_s}^2 u\omega_1 - u\omega_1,$

$\gamma_{[x]}^{[\mu_3]} = 4\omega_2^2 u - 2u\omega_1^2 - 6\omega_2 u - 2\omega_2 u\omega_1 + 6u\omega_1,$

$\gamma_{[t^2]}^{[\mu_1]} = 2 + 6\omega_2 {c_s}^2\omega_1 - \frac{15}{2}\omega_2^2 u^2 - 7u^2\omega_1 - \omega_2 - 2\omega_2 u^2\omega_1^2 + \frac{3}{2}\omega_1^2 + 18\omega_2 u^2\omega_1 - \frac{7}{2}\omega_2 {c_s}^2 - 3u^2\omega_1^2 + \frac{3}{2}\omega_2\omega_1 - 2\omega_2 {c_s}^2\omega_1^2 - 2\omega_2^2 u^2\omega_1 + \frac{3}{2}\omega_2^2 {c_s}^2 - 4\omega_1 - 2\omega_2^2 {c_s}^2\omega_1 + \frac{7}{2}\omega_2 u^2,$

$\gamma_{[t^2]}^{[\mu_3]} = -3 + \frac{9}{2}\omega_2 + 2\omega_2\omega_1^2 - \frac{3}{2}\omega_1^2 - \frac{15}{2}\omega_2\omega_1 + 2\omega_2^2\omega_1 + \frac{9}{2}\omega_1 - \frac{3}{2}\omega_2^2,$

$\gamma_{[tx]}^{[\mu_1]} = 6\omega_2 u^3\omega_1 + 2\omega_2^2 {c_s}^2 u - 4u^3\omega_1^2 - 2\omega_2^2 u^3 - 2\omega_2 {c_s}^2 u\omega_1 - 2\omega_2 u + 2u\omega_1,$

$\gamma_{[tx]}^{[\mu_3]} = -4\omega_2^2 u + 2u\omega_1^2 + 6\omega_2 u + 2\omega_2 u\omega_1 - 6u\omega_1,$

$\gamma_{[x^2]}^{[\mu_1]} = -\frac{1}{2}\omega_2 {c_s}^2\omega_1 + \frac{1}{2}\omega_2^2 u^2 + 3u^2\omega_1 - \frac{1}{2}\omega_2 u^2\omega_1 + \frac{3}{2}\omega_2 {c_s}^2 - u^2\omega_1^2 - \frac{1}{2}\omega_2^2 {c_s}^2 - \frac{3}{2}\omega_2 u^2.$

$\gamma_{[x^2]}^{[\mu_3]} = 1 + 2\omega_2^2 u^2 - \frac{3}{2}\omega_2 - 4\omega_2 u^2\omega_1 + 2u^2\omega_1^2 + \frac{1}{2}\omega_2\omega_1 - \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2^2,$