D1Q3 NSE,
a supplementary material for
Lattice Boltzmann Method Analysis Tool (LBMAT)

Radek Fučík[†], Pavel Eichler[†], Jakub Klinkovský[†], Robert Straka[‡,†], and Tomáš Oberhuber[†]

[†]Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Trojanova 13, 120 00 Prague, Czech Republic
[‡]AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Krakow, Poland

Contents

1 Global definitions

In $\mathbb{R}^1$, the position and velocity vectors are given by $\boldsymbol{x} = (x_1)$ and $\boldsymbol{v} = (v_1)$, respectively.

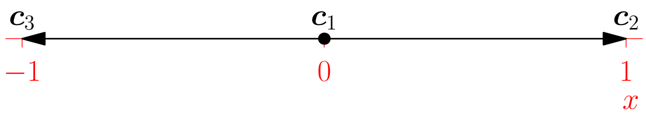
1.1 Discrete velocity vectors

Discrete velocity vectors and the lattice speed of sound are defined by

$$\{\boldsymbol{c}_i\}_{i=1}^{3} = \left(\left(0\right), \left(1\right), \left(-1\right)\right),$$

$$c_s = \frac{1}{\sqrt{3}},$$

respectively [1].



1.2 Raw and central moments

The raw and central moments are defined by

$$m_{\boldsymbol{\alpha}} := \sum_{i=1}^{3} f_i \boldsymbol{c}_i^{\boldsymbol{\alpha}},$$

and

$$k_{\boldsymbol{\alpha}} := \sum_{i=1}^{3} f_i (\boldsymbol{c}_i - \boldsymbol{v})^{\boldsymbol{\alpha}},$$

respectively, where $\boldsymbol{\alpha} = (\alpha_1) \in \mathbb{Z}^1$ denotes a multi-index and $\boldsymbol{c}_i^{\boldsymbol{\alpha}} := [\boldsymbol{c}_i]_1^{\alpha_1}$.

1.3 Transformation matrix M

Matrix $\mathbf{M}$, that defines macroscopic quantities (moments) $\boldsymbol{\mu}$ by

$$\boldsymbol{\mu} = \mathbf{M}\boldsymbol{f},$$

with $\boldsymbol{f} = (f_1, f_2, f_3)^T$, is selected such that

$$\boldsymbol{\mu} = \left(m_{(0)}, m_{(1)}, m_{(2)}\right)^T,$$

i.e., $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

1.4 Equilibrium

The corresponding equilibrium raw moments are defined using the continuous Maxwell–Boltzmann distribution function [1]

$$f^{(eq)}(\xi) = \frac{\rho}{(2\pi c_s^2)^{\frac{1}{2}}} \exp\left(-\frac{(\xi - v_1)^2}{2c_s^2}\right)$$

as

$$m_{(\alpha)}^{(eq)} = \int\limits_{\mathbb{R}} \xi^\alpha f^{(eq)}(\xi) \mathrm{d}\xi,$$

where $\alpha \in \{0, 1, 2\}$. Hence, the equilibrium moments $\boldsymbol{\mu}^{(eq)}$ satisfy

$$\boldsymbol{\mu}^{(eq)} = \left(\rho, \rho v_1, \rho(v_1^2 + c_s^2)\right)^T.$$

2 Spatial EPDEs

2.1 SRT

2.1.1 Definitions

Collision operator $\boldsymbol{C}$:

$$\boldsymbol{C}(\boldsymbol{f}) = \omega\left(\mathbf{M}^{-1}\boldsymbol{\mu}^{(eq)} - \boldsymbol{f}\right),$$

$\omega \in (0, 2)$.

2.1.2 Conservation of mass equation

$$\frac{\partial \rho}{\partial t} + \frac{v_1\delta_l}{\delta_t}\frac{\partial \rho}{\partial x_1} + \frac{\rho\delta_l}{\delta_t}\frac{\partial v_1}{\partial x_1} + (-1 + 3c_s^2 + v_1^2)\frac{v_1\delta_l^3}{12\delta_t}\frac{\partial^3 \rho}{\partial x_1^3} + (-1 + c_s^2 + 3v_1^2)\frac{\rho\delta_l^3}{12\delta_t}\frac{\partial^3 v_1}{\partial x_1^3} +$$
$$(-2c_s^2 - c_s^4\omega + 6v_1^4 - 3\omega v_1^4 - 12c_s^2\omega v_1^2 + c_s^2\omega - 6v_1^2 + 3\omega v_1^2 + 24c_s^2 v_1^2 + 2c_s^4)\frac{\delta_l^4}{24\omega\delta_t}\frac{\partial^4 \rho}{\partial x_1^4} +$$
$$(-4 + 6c_s^2 + 2\omega - 3c_s^2\omega + 10v_1^2 - 5\omega v_1^2)\frac{\rho v_1\delta_l^4}{12\omega\delta_t}\frac{\partial^4 v_1}{\partial x_1^4} = 0.$$

2.1.3 Conservation of momentum equation

$$v_1\frac{\partial \rho}{\partial t} + \rho\frac{\partial v_1}{\partial t} + (c_s^2 + v_1^2)\frac{\delta_l}{\delta_t}\frac{\partial \rho}{\partial x_1} + \frac{2\rho v_1\delta_l}{\delta_t}\frac{\partial v_1}{\partial x_1} + (-2 + 4c_s^2 + \omega - 2c_s^2\omega + 6v_1^2 - 3\omega v_1^2)\frac{\delta_l^2}{\omega\delta_t}\frac{\partial \rho}{\partial x_1}\frac{\partial v_1}{\partial x_1} +$$
$$(2 - \omega)\frac{3\rho v_1\delta_l^2}{\omega\delta_t}\left(\frac{\partial v_1}{\partial x_1}\right)^2 + (-2 + 6c_s^2 + \omega - 3c_s^2\omega + 2v_1^2 - \omega v_1^2)\frac{v_1\delta_l^2}{2\omega\delta_t}\frac{\partial^2 \rho}{\partial x_1^2} + (-2 + 2c_s^2 + \omega - c_s^2\omega + 6v_1^2 - 3\omega v_1^2)\frac{\rho\delta_l^2}{2\omega\delta_t}\frac{\partial^2 v_1}{\partial x_1^2}$$
$$+ C_1\frac{\delta_l^3}{12\omega^2\delta_t}\frac{\partial^3 \rho}{\partial x_1^3} + (-24 + 36c_s^2 + 11\omega^2 v_1^2 + 24\omega - 36c_s^2\omega + 60v_1^2 - 60\omega v_1^2 + 5c_s^2\omega^2 - 4\omega^2)\frac{\rho v_1\delta_l^3}{6\omega^2\delta_t}\frac{\partial^3 v_1}{\partial x_1^3} +$$
$$C_2\frac{v_1\delta_l^4}{12\omega^3\delta_t}\frac{\partial^4 \rho}{\partial x_1^4} + C_3\frac{\rho\delta_l^4}{12\omega^3\delta_t}\frac{\partial^4 v_1}{\partial x_1^4} = 0,$$

where:

$C_1 = -12c_s^2 - 12c_s^4\omega - 7\omega^2 v_1^2 + c_s^4\omega^2 + 24c_s^2\omega^2 v_1^2 + 36v_1^4 - 36\omega v_1^4 - 144c_s^2\omega v_1^2 + 7\omega^2 v_1^4 + 12c_s^2\omega - 36v_1^2 + 36\omega v_1^2 + 144c_s^2 v_1^2 + 12c_s^4 - c_s^2\omega^2$

$C_2 = 12 - 132c_s^2 + 10\omega^3 v_1^2 - 34c_s^2\omega^3 v_1^2 - 216c_s^4\omega - 98\omega^2 v_1^2 + 82c_s^4\omega^2 - 18\omega - 5c_s^4\omega^3 + 404c_s^2\omega^2 v_1^2 + 144v_1^4 - 216\omega v_1^4 - 1008c_s^2\omega v_1^2 + 90\omega^2 v_1^4 + 198c_s^2\omega - 156v_1^2 + 234\omega v_1^2 + 672c_s^2 v_1^2 + 6c_s^2\omega^3 - 9\omega^3 v_1^4 - \omega^3 + 144c_s^4 - 78c_s^2\omega^2 + 8\omega^2$

$C_3 = 12 - 36c_s^2 + 14\omega^3 v_1^2 - 18c_s^2\omega^3 v_1^2 - 36c_s^4\omega - 154\omega^2 v_1^2 + 14c_s^4\omega^2 - 18\omega - c_s^4\omega^3 + 252c_s^2\omega^2 v_1^2 + 504v_1^4 - 756\omega v_1^4 - 648c_s^2\omega v_1^2 + 310\omega^2 v_1^4 + 54c_s^2\omega - 252v_1^2 + 378\omega v_1^2 + 432c_s^2 v_1^2 + 2c_s^2\omega^3 - 29\omega^3 v_1^4 - \omega^3 + 24c_s^4 - 22c_s^2\omega^2 + 8\omega^2$

2.2 MRT

2.2.1 Definitions

Collision operator $\boldsymbol{C}$:

$$\boldsymbol{C}\left(\boldsymbol{f}\right) = \mathbf{M}^{-1}\mathbf{S}\left(\boldsymbol{\mu}^{(eq)} - \mathbf{M}\boldsymbol{f}\right),$$

where

$$\mathbf{S} = \mathrm{diag}(\omega_1, \omega_2, \omega_3),$$

$\omega_1, \omega_2, \omega_3 \in (0, 2)$.

2.2.2 Conservation of mass equation

$$\frac{\partial \rho}{\partial t} + \frac{\delta_l v_1}{\delta_t}\frac{\partial \rho}{\partial x_1} + \frac{\delta_l \rho}{\delta_t}\frac{\partial v_1}{\partial x_1} + (-1 + 3c_s^2 + v_1^2)\frac{\delta_l^3 v_1}{12\delta_t}\frac{\partial^3 \rho}{\partial x_1^3} + (-1 + c_s^2 + 3v_1^2)\frac{\delta_l^3 \rho}{12\delta_t}\frac{\partial^3 v_1}{\partial x_1^3} + (3\omega_3 v_1^2 + 24c_s^2 v_1^2 - c_s^4\omega_3 - 12c_s^2\omega_3 v_1^2 - 2c_s^2 + 6v_1^4 + 2c_s^4 - 6v_1^2 - 3\omega_3 v_1^4 + c_s^2\omega_3)\frac{\delta_l^4}{24\omega_3\delta_t}\frac{\partial^4 \rho}{\partial x_1^4} + (-4 - 5\omega_3 v_1^2 + 6c_s^2 + 10v_1^2 + 2\omega_3 - 3c_s^2\omega_3)\frac{\delta_l^4 \rho v_1}{12\omega_3\delta_t}\frac{\partial^4 v_1}{\partial x_1^4} = 0.$$

2.2.3 Conservation of momentum equation

$$v_1\frac{\partial \rho}{\partial t} + \rho\frac{\partial v_1}{\partial t} + (c_s^2 + v_1^2)\frac{\delta_l}{\delta_t}\frac{\partial \rho}{\partial x_1} + \frac{2\delta_l \rho v_1}{\delta_t}\frac{\partial v_1}{\partial x_1} + (-2 - 3\omega_3 v_1^2 + 4c_s^2 + 6v_1^2 + \omega_3 - 2c_s^2\omega_3)\frac{\delta_l^2}{\omega_3\delta_t}\frac{\partial \rho}{\partial x_1}\frac{\partial v_1}{\partial x_1} + (2 - \omega_3)\frac{3\delta_l^2 \rho v_1}{\omega_3\delta_t}\left(\frac{\partial v_1}{\partial x_1}\right)^2 + (-2 - \omega_3 v_1^2 + 6c_s^2 + 2v_1^2 + \omega_3 - 3c_s^2\omega_3)\frac{\delta_l^2 v_1}{2\omega_3\delta_t}\frac{\partial^2 \rho}{\partial x_1^2} + (-2 - 3\omega_3 v_1^2 + 2c_s^2 + 6v_1^2 + \omega_3 - c_s^2\omega_3)\frac{\delta_l^2 \rho}{2\omega_3\delta_t}\frac{\partial^2 v_1}{\partial x_1^2} + C_1\frac{\delta_l^3}{12\omega_3^2\delta_t}\frac{\partial^3 \rho}{\partial x_1^3} + (-24 - 60\omega_3 v_1^2 - 4\omega_3^2 + 36c_s^2 + 5c_s^2\omega_3^2 + 60v_1^2 + 24\omega_3 - 36c_s^2\omega_3 + 11\omega_3^2 v_1^2)\frac{\delta_l^3 \rho v_1}{6\omega_3^2\delta_t}\frac{\partial^3 v_1}{\partial x_1^3} + C_2\frac{\delta_l^4 v_1}{12\omega_3^3\delta_t}\frac{\partial^4 \rho}{\partial x_1^4} + C_3\frac{\delta_l^4 \rho}{12\omega_3^3\delta_t}\frac{\partial^4 v_1}{\partial x_1^4} = 0,$$

where:

$C_1 = 36\omega_3 v_1^2 + 7\omega_3^2 v_1^4 + 144c_s^2 v_1^2 + c_s^4\omega_3^2 - 12c_s^4\omega_3 - 144c_s^2\omega_3 v_1^2 - 12c_s^2 + 36v_1^4 - c_s^2\omega_3^2 + 12c_s^4 - 36v_1^2 + 24c_s^2\omega_3^2 v_1^2 - 36\omega_3 v_1^4 + 12c_s^2\omega_3 - 7\omega_3^2 v_1^2$

$C_2 = 12 + 234\omega_3 v_1^2 - 5c_s^4\omega_3^3 + 8\omega_3^2 + 90\omega_3^2 v_1^4 + 672c_s^2 v_1^2 - \omega_3^3 + 82c_s^4\omega_3^2 - 216c_s^4\omega_3 - 1008c_s^2\omega_3 v_1^2 - 132c_s^2 + 144v_1^4 - 9\omega_3^3 v_1^4 - 78c_s^2\omega_3^2 + 144c_s^4 - 156v_1^2 + 10\omega_3^3 v_1^2 - 18\omega_3 + 6c_s^2\omega_3^3 + 404c_s^2\omega_3^2 v_1^2 - 216\omega_3 v_1^4 - 34c_s^2\omega_3^3 v_1^2 + 198c_s^2\omega_3 - 98\omega_3^2 v_1^2$

$C_3 = 12 + 378\omega_3 v_1^2 - c_s^4\omega_3^3 + 8\omega_3^2 + 310\omega_3^2 v_1^4 + 432c_s^2 v_1^2 - \omega_3^3 + 14c_s^4\omega_3^2 - 36c_s^4\omega_3 - 648c_s^2\omega_3 v_1^2 - 36c_s^2 + 504v_1^4 - 29\omega_3^3 v_1^4 - 22c_s^2\omega_3^2 + 24c_s^4 - 252v_1^2 + 14\omega_3^3 v_1^2 - 18\omega_3 + 2c_s^2\omega_3^3 + 252c_s^2\omega_3^2 v_1^2 - 756\omega_3 v_1^4 - 18c_s^2\omega_3^3 v_1^2 + 54c_s^2\omega_3 - 154\omega_3^2 v_1^2$

2.3 CLBM

2.3.1 Definitions

Collision operator $\boldsymbol{C}$:

$$\boldsymbol{C}\left(\boldsymbol{f}\right) = \mathbf{K}^{-1}\mathbf{S}\left(\boldsymbol{\kappa}^{(eq)} - \mathbf{K}\boldsymbol{f}\right),$$

where

$$\mathbf{S} = \mathrm{diag}(\omega_1, \omega_2, \omega_3),$$

$\omega_1, \omega_2, \omega_3 \in (0, 2)$.
Matrix $\mathbf{K}$ corresponds to the transformation matrix to the central moment basis defined by

$$\boldsymbol{\kappa} = \left(k_{(0)}, k_{(1)}, k_{(2)}\right)^T$$

and is given by

$$\mathbf{K} = \begin{pmatrix} 1 & 1 & 1 \\ -v_1 & 1 - v_1 & -v_1 - 1 \\ v_1^2 & (1 - v_1)^2 & (v_1 + 1)^2 \end{pmatrix}.$$

The equilibrium central moments are defined by

$$\boldsymbol{\kappa}^{(eq)} = \mathbf{K}\mathbf{M}^{-1}\boldsymbol{\mu}^{(eq)},$$

i.e.,

$$\boldsymbol{\kappa}^{(eq)} = \left(\rho, 0, \rho c_s^2\right)^T.$$

2.3.2 Conservation of mass equation

$$\frac{\partial \rho}{\partial t} + \frac{\delta_l v_1}{\delta_t}\frac{\partial \rho}{\partial x_1} + \frac{\rho \delta_l}{\delta_t}\frac{\partial v_1}{\partial x_1} + (-1 + 3c_s^2 + v_1^2)\frac{\delta_l^3 v_1}{12\delta_t}\frac{\partial^3 \rho}{\partial x_1^3} + (-1 + c_s^2 + 3v_1^2)\frac{\rho \delta_l^3}{12\delta_t}\frac{\partial^3 v_1}{\partial x_1^3} +$$
$$(\omega_3 c_s^2 - 2c_s^2 - 12\omega_3 v_1^2 c_s^2 + 6v_1^4 - 3\omega_3 v_1^4 + 24v_1^2 c_s^2 + 3\omega_3 v_1^2 - 6v_1^2 + 2c_s^4 - \omega_3 c_s^4)\frac{\delta_l^4}{24\omega_3 \delta_t}\frac{\partial^4 \rho}{\partial x_1^4} +$$
$$(-4 - 3\omega_3 c_s^2 + 6c_s^2 + 2\omega_3 - 5\omega_3 v_1^2 + 10v_1^2)\frac{\rho \delta_l^4 v_1}{12\omega_3 \delta_t}\frac{\partial^4 v_1}{\partial x_1^4} = 0.$$

2.3.3 Conservation of momentum equation

$$v_1 \frac{\partial \rho}{\partial t} + \rho \frac{\partial v_1}{\partial t} + (c_s^2 + v_1^2)\frac{\delta_l}{\delta_t}\frac{\partial \rho}{\partial x_1} + \frac{2\rho \delta_l v_1}{\delta_t}\frac{\partial v_1}{\partial x_1} + (-2 - 2\omega_3 c_s^2 + 4c_s^2 + \omega_3 - 3\omega_3 v_1^2 + 6v_1^2)\frac{\delta_l^2}{\omega_3 \delta_t}\frac{\partial \rho}{\partial x_1}\frac{\partial v_1}{\partial x_1} +$$
$$(2 - \omega_3)\frac{3\rho \delta_l^2 v_1}{\omega_3 \delta_t}\left(\frac{\partial v_1}{\partial x_1}\right)^2 + (-2 - 3\omega_3 c_s^2 + 6c_s^2 + \omega_3 - \omega_3 v_1^2 + 2v_1^2)\frac{\delta_l^2 v_1}{2\omega_3 \delta_t}\frac{\partial^2 \rho}{\partial x_1^2} +$$
$$(-2 - \omega_3 c_s^2 + 2c_s^2 + \omega_3 - 3\omega_3 v_1^2 + 6v_1^2)\frac{\rho \delta_l^2}{2\omega_3 \delta_t}\frac{\partial^2 v_1}{\partial x_1^2} + C_1 \frac{\delta_l^3}{12\omega_3^2 \delta_t}\frac{\partial^3 \rho}{\partial x_1^3} +$$
$$(-24 - 36\omega_3 c_s^2 + 36c_s^2 + 11\omega_3^2 v_1^2 + 24\omega_3 - 60\omega_3 v_1^2 + 60v_1^2 + 5\omega_3^2 c_s^2 - 4\omega_3^2)\frac{\rho \delta_l^3 v_1}{6\omega_3^2 \delta_t}\frac{\partial^3 v_1}{\partial x_1^3} + C_2 \frac{\delta_l^4 v_1}{12\omega_3^3 \delta_t}\frac{\partial^4 \rho}{\partial x_1^4} +$$
$$C_3 \frac{\rho \delta_l^4}{12\omega_3^3 \delta_t}\frac{\partial^4 v_1}{\partial x_1^4} = 0,$$

where:

$$C_1 = \omega_3^2 c_s^4 + 12\omega_3 c_s^2 - 12c_s^2 - 144\omega_3 v_1^2 c_s^2 - 7\omega_3^2 v_1^2 + 36v_1^4 - 36\omega_3 v_1^4 + 144v_1^2 c_s^2 + 24\omega_3^2 v_1^2 c_s^2 + 7\omega_3^2 v_1^4 + 36\omega_3 v_1^2 - 36v_1^2 - \omega_3^2 c_s^2 + 12c_s^4 - 12\omega_3 c_s^4$$

$$C_2 = 12 + 82\omega_3^2 c_s^4 + 10\omega_3^3 v_1^2 + 198\omega_3 c_s^2 - 132c_s^2 - 34\omega_3^3 v_1^2 c_s^2 - 1008\omega_3 v_1^2 c_s^2 - 5\omega_3^3 c_s^4 - 98\omega_3^2 v_1^2 - 18\omega_3 + 144v_1^4 - 216\omega_3 v_1^4 + 672v_1^2 c_s^2 +$$
$$404\omega_3^2 v_1^2 c_s^2 + 6\omega_3^3 c_s^2 + 90\omega_3^2 v_1^4 + 234\omega_3 v_1^2 - 156v_1^2 - \omega_3^3 - 78\omega_3^2 c_s^2 - 9\omega_3^3 v_1^4 + 144c_s^4 - 216\omega_3 c_s^4 + 8\omega_3^2$$

$$C_3 = 12 + 14\omega_3^2 c_s^4 + 14\omega_3^3 v_1^2 + 54\omega_3 c_s^2 - 36c_s^2 - 18\omega_3^3 v_1^2 c_s^2 - 648\omega_3 v_1^2 c_s^2 - \omega_3^3 c_s^4 - 154\omega_3^2 v_1^2 - 18\omega_3 + 504v_1^4 - 756\omega_3 v_1^4 + 432v_1^2 c_s^2 +$$
$$252\omega_3^2 v_1^2 c_s^2 + 2\omega_3^3 c_s^2 + 310\omega_3^2 v_1^4 + 378\omega_3 v_1^2 - 252v_1^2 - \omega_3^3 - 22\omega_3^2 c_s^2 - 29\omega_3^3 v_1^4 + 24c_s^4 - 36\omega_3 c_s^4 + 8\omega_3^2$$

3 Comparison of SRT, MRT, and CLBM

3.1 Conservation of mass equation

$$\frac{\partial\rho}{\partial t} + v_1\frac{\delta_l}{\delta_t}\frac{\partial\rho}{\partial x_1} + \rho\frac{\delta_l}{\delta_t}\frac{\partial v_1}{\partial x_1} + (-1+3c_s^2+v_1^2)\frac{v_1}{12}\frac{\delta_l^3}{\delta_t}\frac{\partial^3\rho}{\partial x_1^3} + (-1+c_s^2+3v_1^2)\frac{\rho}{12}\frac{\delta_l^3}{\delta_t}\frac{\partial^3 v_1}{\partial x_1^3} + C^{(0)}_{\mathrm{D}_x^4\rho}\frac{\delta_l^4}{\delta_t}\frac{\partial^4\rho}{\partial x_1^4} + C^{(0)}_{\mathrm{D}_x^4 v_1}\frac{\delta_l^4}{\delta_t}\frac{\partial^4 v_1}{\partial x_1^4} = 0,$$

where:

coefficient $C^{(0)}_{\mathrm{D}_x^4\rho}$ **at** $\frac{\partial^4\rho}{\partial x_1^4}$**:**

$C^{(0),\mathrm{SRT}}_{\mathrm{D}_x^4\rho} = (6v_1^4+3\omega v_1^2-\omega c_s^4-2c_s^2-6v_1^2-12\omega v_1^2c_s^2-3\omega v_1^4+\omega c_s^2+24v_1^2c_s^2+2c_s^4)\frac{1}{24\omega}$

$C^{(0),\mathrm{MRT1}}_{\mathrm{D}_x^4\rho} = (6v_1^4-3\omega_3 v_1^4-2c_s^2+\omega_3 c_s^2-12\omega_3 v_1^2c_s^2+3\omega_3 v_1^2-6v_1^2-\omega_3 c_s^4+24v_1^2c_s^2+2c_s^4)\frac{1}{24\omega_3}$

$C^{(0),\mathrm{CLBM1}}_{\mathrm{D}_x^4\rho} = C^{(0),\mathrm{MRT1}}_{\mathrm{D}_x^4\rho}$

coefficient $C^{(0)}_{\mathrm{D}_x^4 v_1}$ **at** $\frac{\partial^4 v_1}{\partial x_1^4}$**:**

$C^{(0),\mathrm{SRT}}_{\mathrm{D}_x^4 v_1} = (-4-5\omega v_1^2+2\omega+6c_s^2+10v_1^2-3\omega c_s^2)\frac{v_1\rho}{12\omega}$

$C^{(0),\mathrm{MRT1}}_{\mathrm{D}_x^4 v_1} = (-4+6c_s^2-3\omega_3 c_s^2+2\omega_3-5\omega_3 v_1^2+10v_1^2)\frac{v_1\rho}{12\omega_3}$

$C^{(0),\mathrm{CLBM1}}_{\mathrm{D}_x^4 v_1} = C^{(0),\mathrm{MRT1}}_{\mathrm{D}_x^4 v_1}$

3.2 Conservation of momentum equation

$$v_1\frac{\partial\rho}{\partial t} + \rho\frac{\partial v_1}{\partial t} + (c_s^2+v_1^2)\frac{\delta_l}{\delta_t}\frac{\partial\rho}{\partial x_1} + 2v_1\rho\frac{\delta_l}{\delta_t}\frac{\partial v_1}{\partial x_1} + C^{(1)}_{\mathrm{D}_x\rho,\mathrm{D}_x v_1}\frac{\delta_l^2}{\delta_t}\frac{\partial\rho}{\partial x_1}\frac{\partial v_1}{\partial x_1} + C^{(1)}_{\mathrm{D}_x v_1,\mathrm{D}_x v_1}\frac{\delta_l^2}{\delta_t}\left(\frac{\partial v_1}{\partial x_1}\right)^2 + C^{(1)}_{\mathrm{D}_x^2\rho}\frac{\delta_l^2}{\delta_t}\frac{\partial^2\rho}{\partial x_1^2} + C^{(1)}_{\mathrm{D}_x^2 v_1}\frac{\delta_l^2}{\delta_t}\frac{\partial^2 v_1}{\partial x_1^2} + C^{(1)}_{\mathrm{D}_x^3\rho}\frac{\delta_l^3}{\delta_t}\frac{\partial^3\rho}{\partial x_1^3} + C^{(1)}_{\mathrm{D}_x^3 v_1}\frac{\delta_l^3}{\delta_t}\frac{\partial^3 v_1}{\partial x_1^3} + C^{(1)}_{\mathrm{D}_x^4\rho}\frac{\delta_l^4}{\delta_t}\frac{\partial^4\rho}{\partial x_1^4} + C^{(1)}_{\mathrm{D}_x^4 v_1}\frac{\delta_l^4}{\delta_t}\frac{\partial^4 v_1}{\partial x_1^4} = 0,$$

where:

coefficient $C^{(1)}_{\mathrm{D}_x\rho,\mathrm{D}_x v_1}$ **at** $\frac{\partial\rho}{\partial x_1}\frac{\partial v_1}{\partial x_1}$**:**

$C^{(1),\mathrm{SRT}}_{\mathrm{D}_x\rho,\mathrm{D}_x v_1} = (-2-3\omega v_1^2+\omega+4c_s^2+6v_1^2-2\omega c_s^2)\frac{1}{\omega}$

$C^{(1),\mathrm{MRT1}}_{\mathrm{D}_x\rho,\mathrm{D}_x v_1} = (-2+4c_s^2-2\omega_3 c_s^2+\omega_3-3\omega_3 v_1^2+6v_1^2)\frac{1}{\omega_3}$

$C^{(1),\mathrm{CLBM1}}_{\mathrm{D}_x\rho,\mathrm{D}_x v_1} = C^{(1),\mathrm{MRT1}}_{\mathrm{D}_x\rho,\mathrm{D}_x v_1}$

coefficient $C^{(1)}_{\mathrm{D}_x v_1,\mathrm{D}_x v_1}$ **at** $\left(\frac{\partial v_1}{\partial x_1}\right)^2$**:**

$C^{(1),\mathrm{SRT}}_{\mathrm{D}_x v_1,\mathrm{D}_x v_1} = (2-\omega)\frac{3v_1\rho}{\omega}$

$C^{(1),\mathrm{MRT1}}_{\mathrm{D}_x v_1,\mathrm{D}_x v_1} = (2-\omega_3)\frac{3v_1\rho}{\omega_3}$

$C^{(1),\mathrm{CLBM1}}_{\mathrm{D}_x v_1,\mathrm{D}_x v_1} = C^{(1),\mathrm{MRT1}}_{\mathrm{D}_x v_1,\mathrm{D}_x v_1}$

coefficient $C^{(1)}_{\mathrm{D}_x^2\rho}$ **at** $\frac{\partial^2\rho}{\partial x_1^2}$**:**

$C^{(1),\mathrm{SRT}}_{\mathrm{D}_x^2\rho} = (-2-\omega v_1^2+\omega+6c_s^2+2v_1^2-3\omega c_s^2)\frac{v_1}{2\omega}$

$C^{(1),\mathrm{MRT1}}_{\mathrm{D}_x^2\rho} = (-2+6c_s^2-3\omega_3 c_s^2+\omega_3-\omega_3 v_1^2+2v_1^2)\frac{v_1}{2\omega_3}$

$C^{(1),\mathrm{CLBM1}}_{\mathrm{D}^2_x\rho} = C^{(1),\mathrm{MRT1}}_{\mathrm{D}^2_x\rho}$

coefficient $C^{(1)}_{\mathrm{D}^2_x v_1}$ **at** $\frac{\partial^2 v_1}{\partial x_1^2}$**:**

$C^{(1),\mathrm{SRT}}_{\mathrm{D}^2_x v_1} = (-2 - 3\omega v_1^2 + \omega + 2c_s^2 + 6v_1^2 - \omega c_s^2)\frac{\rho}{2\omega}$

$C^{(1),\mathrm{MRT1}}_{\mathrm{D}^2_x v_1} = (-2 + 2c_s^2 - \omega_3 c_s^2 + \omega_3 - 3\omega_3 v_1^2 + 6v_1^2)\frac{\rho}{2\omega_3}$

$C^{(1),\mathrm{CLBM1}}_{\mathrm{D}^2_x v_1} = C^{(1),\mathrm{MRT1}}_{\mathrm{D}^2_x v_1}$

coefficient $C^{(1)}_{\mathrm{D}^3_x\rho}$ **at** $\frac{\partial^3 \rho}{\partial x_1^3}$**:**

$C^{(1),\mathrm{SRT}}_{\mathrm{D}^3_x\rho} = (36v_1^4 + 36\omega v_1^2 - \omega^2 c_s^2 + 24\omega^2 v_1^2 c_s^2 - 12\omega c_s^4 - 12c_s^2 + 7\omega^2 v_1^4 + \omega^2 c_s^4 - 36v_1^2 - 144\omega v_1^2 c_s^2 - 36\omega v_1^4 - 7\omega^2 v_1^2 + 12\omega c_s^2 + 144v_1^2 c_s^2 + 12c_s^4)\frac{1}{12\omega^2}$

$C^{(1),\mathrm{MRT1}}_{\mathrm{D}^3_x\rho} =$
$(\omega_3^2 c_s^4 + 36v_1^4 - 36\omega_3 v_1^4 - 7\omega_3^2 v_1^2 - 12c_s^2 + 12\omega_3 c_s^2 - 144\omega_3 v_1^2 c_s^2 + 24\omega_3^2 v_1^2 c_s^2 + 36\omega_3 v_1^2 - \omega_3^2 c_s^2 - 36v_1^2 - 12\omega_3 c_s^4 + 144v_1^2 c_s^2 + 7\omega_3^2 v_1^4 + 12c_s^4)\frac{1}{12\omega_3^2}$

$C^{(1),\mathrm{CLBM1}}_{\mathrm{D}^3_x\rho} = C^{(1),\mathrm{MRT1}}_{\mathrm{D}^3_x\rho}$

coefficient $C^{(1)}_{\mathrm{D}^3_x v_1}$ **at** $\frac{\partial^3 v_1}{\partial x_1^3}$**:**

$C^{(1),\mathrm{SRT}}_{\mathrm{D}^3_x v_1} = (-24 - 60\omega v_1^2 + 24\omega + 5\omega^2 c_s^2 + 36c_s^2 - 4\omega^2 + 60v_1^2 + 11\omega^2 v_1^2 - 36\omega c_s^2)\frac{v_1\rho}{6\omega^2}$

$C^{(1),\mathrm{MRT1}}_{\mathrm{D}^3_x v_1} = (-24 + 11\omega_3^2 v_1^2 + 36c_s^2 - 36\omega_3 c_s^2 - 4\omega_3^2 + 24\omega_3 - 60\omega_3 v_1^2 + 5\omega_3^2 c_s^2 + 60v_1^2)\frac{v_1\rho}{6\omega_3^2}$

$C^{(1),\mathrm{CLBM1}}_{\mathrm{D}^3_x v_1} = C^{(1),\mathrm{MRT1}}_{\mathrm{D}^3_x v_1}$

coefficient $C^{(1)}_{\mathrm{D}^4_x\rho}$ **at** $\frac{\partial^4 \rho}{\partial x_1^4}$**:**

$C^{(1),\mathrm{SRT}}_{\mathrm{D}^4_x\rho} = (12 + 144v_1^4 + 234\omega v_1^2 - 18\omega - 78\omega^2 c_s^2 + 404\omega^2 v_1^2 c_s^2 - 216\omega c_s^4 - 132c_s^2 + 90\omega^2 v_1^4 + 6\omega^3 c_s^2 - 9\omega^3 v_1^4 - \omega^3 - 5\omega^3 c_s^4 + 8\omega^2 -$
$34\omega^3 v_1^2 c_s^2 + 10\omega^3 v_1^2 + 82\omega^2 c_s^4 - 156v_1^2 - 1008\omega v_1^2 c_s^2 - 216\omega v_1^4 - 98\omega^2 v_1^2 + 198\omega c_s^2 + 672v_1^2 c_s^2 + 144c_s^4)\frac{v_1}{12\omega^3}$

$C^{(1),\mathrm{MRT1}}_{\mathrm{D}^4_x\rho} = (12 + 82\omega_3^2 c_s^4 + 144v_1^4 - 216\omega_3 v_1^4 - 34\omega_3^3 v_1^2 c_s^2 - 98\omega_3^2 v_1^2 - 132c_s^2 + 198\omega_3 c_s^2 + 8\omega_3^2 - 5\omega_3^3 c_s^4 - 1008\omega_3 v_1^2 c_s^2 - \omega_3^3 + 10\omega_3^3 v_1^2 -$
$18\omega_3 + 6\omega_3^3 c_s^2 + 404\omega_3^2 v_1^2 c_s^2 - 9\omega_3^3 v_1^4 + 234\omega_3 v_1^2 - 78\omega_3^2 c_s^2 - 156v_1^2 - 216\omega_3 c_s^4 + 672v_1^2 c_s^2 + 90\omega_3^2 v_1^4 + 144c_s^4)\frac{v_1}{12\omega_3^3}$

$C^{(1),\mathrm{CLBM1}}_{\mathrm{D}^4_x\rho} = C^{(1),\mathrm{MRT1}}_{\mathrm{D}^4_x\rho}$

coefficient $C^{(1)}_{\mathrm{D}^4_x v_1}$ **at** $\frac{\partial^4 v_1}{\partial x_1^4}$**:**

$C^{(1),\mathrm{SRT}}_{\mathrm{D}^4_x v_1} = (12 + 504v_1^4 + 378\omega v_1^2 - 18\omega - 22\omega^2 c_s^2 + 252\omega^2 v_1^2 c_s^2 - 36\omega c_s^4 - 36c_s^2 + 310\omega^2 v_1^4 + 2\omega^3 c_s^2 - 29\omega^3 v_1^4 - \omega^3 - \omega^3 c_s^4 + 8\omega^2 -$
$18\omega^3 v_1^2 c_s^2 + 14\omega^3 v_1^2 + 14\omega^2 c_s^4 - 252v_1^2 - 648\omega v_1^2 c_s^2 - 756\omega v_1^4 - 154\omega^2 v_1^2 + 54\omega c_s^2 + 432v_1^2 c_s^2 + 24c_s^4)\frac{\rho}{12\omega^3}$

$C^{(1),\mathrm{MRT1}}_{\mathrm{D}^4_x v_1} = (12 + 14\omega_3^2 c_s^4 + 504v_1^4 - 756\omega_3 v_1^4 - 18\omega_3^3 v_1^2 c_s^2 - 154\omega_3^2 v_1^2 - 36c_s^2 + 54\omega_3 c_s^2 + 8\omega_3^2 - \omega_3^3 c_s^4 - 648\omega_3 v_1^2 c_s^2 - \omega_3^3 + 14\omega_3^3 v_1^2 - 18\omega_3 +$
$2\omega_3^3 c_s^2 + 252\omega_3^2 v_1^2 c_s^2 - 29\omega_3^3 v_1^4 + 378\omega_3 v_1^2 - 22\omega_3^2 c_s^2 - 252v_1^2 - 36\omega_3 c_s^4 + 432v_1^2 c_s^2 + 310\omega_3^2 v_1^4 + 24c_s^4)\frac{\rho}{12\omega_3^3}$

$C^{(1),\mathrm{CLBM1}}_{\mathrm{D}^4_x v_1} = C^{(1),\mathrm{MRT1}}_{\mathrm{D}^4_x v_1}$

References

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