D1Q3 NSE,
a supplementary material for
Lattice Boltzmann Method Analysis Tool (LBMAT)

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Contents

1 Global definitions

In $\mathbb{R}^1$, the position and velocity vectors are given by $\boldsymbol{x} = (x_1)$ and $\boldsymbol{v} = (v_1)$, respectively.
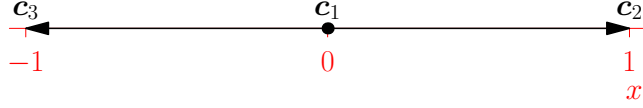
1.1   Discrete velocity vectors

Discrete velocity vectors and the lattice speed of sound are defined by

$$\{\boldsymbol{c}_i\}_{i=1}^3 = \left((0),(1),(-1)\right),$$

$$c_s = \frac{1}{\sqrt{3}},$$

respectively [1].

1.2   Raw and central moments

The raw and central moments are defined by

$$m_{\boldsymbol{\alpha}} := \sum_{i=1}^{3} f_i \boldsymbol{c}_i^{\boldsymbol{\alpha}},$$

and

$$k_{\boldsymbol{\alpha}} := \sum_{i=1}^{3} f_i (\boldsymbol{c}_i - \boldsymbol{v})^{\boldsymbol{\alpha}},$$

respectively, where $\boldsymbol{\alpha} = (\alpha_1) \in \mathbb{Z}^1$ denotes a multi-index and $\boldsymbol{c}_i^{\boldsymbol{\alpha}} := [\boldsymbol{c}_i]_1^{\alpha_1}$.

1.3   Transformation matrix M

Matrix $\mathbf{M}$, that defines macroscopic quantities (moments) $\boldsymbol{\mu}$ by

$$\boldsymbol{\mu} = \mathbf{M}\boldsymbol{f},$$

2

with $\boldsymbol{f} = (f_1, f_2, f_3)^T$, is selected such that

$$\boldsymbol{\mu} = \left(m_{(0)}, m_{(1)}, m_{(2)} \right)^T,$$

i.e., $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

1.4 Equilibrium

The corresponding equilibrium raw moments are defined using the continuous Maxwell–Boltzmann distribution function [1]

$$f^{(eq)}(\xi) = \frac{\rho}{(2\pi c_s^2)^{\frac{1}{2}}} \exp\left(-\frac{(\xi - v_1)^2}{2c_s^2} \right)$$

as

$$m_{(\alpha)}^{(eq)} = \int_{\mathbb{R}} \xi^\alpha f^{(eq)}(\xi)\mathrm{d}\xi,$$

where $\alpha \in \{0, 1, 2\}$. Hence, the equilibrium moments $\boldsymbol{\mu}^{(eq)}$ satisfy

$$\boldsymbol{\mu}^{(eq)} = \left(\rho, \rho v_1, \rho(v_1^2 + c_s^2) \right)^T.$$

2 Spatial EPDEs

2.1 SRT

2.1.1 Definitions

Collision operator $\boldsymbol{C}$:

$$\boldsymbol{C}(\boldsymbol{f}) = \omega \left(\mathbf{M}^{-1} \boldsymbol{\mu}^{(eq)} - \boldsymbol{f} \right),$$

$\omega \in (0, 2)$.

2.1.2 Conservation of mass equation

⏸ attached text file: `output_d1q3_nse_srt_symbolic_pde_00.txt`

$\frac{\partial \rho}{\partial t} + \frac{v_1 \delta_l}{\delta_t} \frac{\partial \rho}{\partial x_1} + \frac{\rho \delta_l}{\delta_t} \frac{\partial v_1}{\partial x_1} + (-1 + v_1^2 + 3c_s^2)\frac{v_1 \delta_l^3}{12\delta_t} \frac{\partial^3 \rho}{\partial x_1^3} + (-1 + 3v_1^2 + c_s^2)\frac{\rho \delta_l^3}{12\delta_t} \frac{\partial^3 v_1}{\partial x_1^3} +$
$(3v_1^2\omega + c_s^2\omega - 6v_1^2 - 2c_s^2 - 12v_1^2 c_s^2\omega + 6v_1^4 + 2c_s^4 + 24v_1^2 c_s^2 - 3v_1^4\omega - c_s^4\omega)\frac{\delta_l^4}{24\delta_t\omega} \frac{\partial^4 \rho}{\partial x_1^4} +$
$(-4 - 5v_1^2\omega - 3c_s^2\omega + 10v_1^2 + 6c_s^2 + 2\omega)\frac{\rho v_1 \delta_l^4}{12\delta_t\omega} \frac{\partial^4 v_1}{\partial x_1^4} = 0.$

2.1.3 Conservation of momentum equation

⏸ attached text file: `output_d1q3_nse_srt_symbolic_pde_01.txt`

$v_1 \frac{\partial \rho}{\partial t} + \rho \frac{\partial v_1}{\partial t} + (v_1^2 + c_s^2) \frac{\delta_l}{\delta_t} \frac{\partial \rho}{\partial x_1} + \frac{2\rho v_1 \delta_l}{\delta_t} \frac{\partial v_1}{\partial x_1} + (-2 - 3v_1^2 \omega - 2c_s^2 \omega + 6v_1^2 + 4c_s^2 + \omega) \frac{\delta_l^2}{\delta_t \omega} \frac{\partial \rho}{\partial x_1} \frac{\partial v_1}{\partial x_1} +$

$(2 - \omega) \frac{3\rho v_1 \delta_l^2}{\delta_t \omega} \left(\frac{\partial v_1}{\partial x_1}\right)^2 + (-2 - v_1^2 \omega - 3c_s^2 \omega + 2v_1^2 + 6c_s^2 + \omega) \frac{v_1 \delta_l^2}{2\delta_t \omega} \frac{\partial^2 \rho}{\partial x_1^2} + (-2 - 3v_1^2 \omega - c_s^2 \omega + 6v_1^2 + 2c_s^2 + \omega) \frac{\rho \delta_l^2}{2\delta_t \omega} \frac{\partial^2 v_1}{\partial x_1^2}$

$+ \; C_1 \frac{\delta_l^3}{12\delta_t \omega^2} \frac{\partial^3 \rho}{\partial x_1^3} + (-24 + 5c_s^2 \omega^2 + 11v_1^2 \omega^2 - 60v_1^2 \omega - 36c_s^2 \omega + 60v_1^2 + 36c_s^2 + 24\omega - 4\omega^2) \frac{\rho v_1 \delta_l^3}{6\delta_t \omega^2} \frac{\partial^3 v_1}{\partial x_1^3} +$

$C_2 \frac{v_1 \delta_l^4}{12\delta_t \omega^3} \frac{\partial^4 \rho}{\partial x_1^4} + \; C_3 \frac{\rho \delta_l^4}{12\delta_t \omega^3} \frac{\partial^4 v_1}{\partial x_1^4} = 0,$

where:

$C_1 = 24v_1^2 c_s^2 \omega^2 - c_s^2 \omega^2 - 7v_1^2 \omega^2 + 36v_1^2 \omega + 12c_s^2 \omega - 36v_1^2 - 12c_s^2 - 144v_1^2 c_s^2 \omega + c_s^4 \omega^2 + 7v_1^4 \omega^2 + 36v_1^4 + 12c_s^4 + 144v_1^2 c_s^2 - 36v_1^4 \omega - 12c_s^4 \omega$

$C_2 = 12 + 6c_s^2 \omega^3 + 404v_1^2 c_s^2 \omega^2 + 10v_1^2 \omega^3 - 34v_1^2 c_s^2 \omega^3 - 78c_s^2 \omega^2 - 98v_1^2 \omega^2 + 234v_1^2 \omega + 198c_s^2 \omega - 156v_1^2 - 132c_s^2 - 1008v_1^2 c_s^2 \omega + 82c_s^4 \omega^2 + 90v_1^4 \omega^2 - 18\omega + 144v_1^4 - 5c_s^4 \omega^3 + 144c_s^4 + 672v_1^2 c_s^2 - 9v_1^4 \omega^3 - \omega^3 - 216v_1^4 \omega + 8\omega^2 - 216c_s^4 \omega$

$C_3 = 12 + 2c_s^2 \omega^3 + 252v_1^2 c_s^2 \omega^2 + 14v_1^2 \omega^3 - 18v_1^2 c_s^2 \omega^3 - 22c_s^2 \omega^2 - 154v_1^2 \omega^2 + 378v_1^2 \omega + 54c_s^2 \omega - 252v_1^2 - 36c_s^2 - 648v_1^2 c_s^2 \omega + 14c_s^4 \omega^2 + 310v_1^4 \omega^2 - 18\omega + 504v_1^4 - c_s^4 \omega^3 + 24c_s^4 + 432v_1^2 c_s^2 - 29v_1^4 \omega^3 - \omega^3 - 756v_1^4 \omega + 8\omega^2 - 36c_s^4 \omega$

2.2 MRT

2.2.1 Definitions

Collision operator $\boldsymbol{C}$:

$$\boldsymbol{C}\left(\boldsymbol{f}\right) = \mathbf{M}^{-1}\mathbf{S}\left(\boldsymbol{\mu}^{(eq)} - \mathbf{M}\boldsymbol{f}\right),$$

where

$$\mathbf{S} = \mathrm{diag}(\omega_1, \omega_2, \omega_3),$$

$\omega_1, \omega_2, \omega_3 \in (0, 2)$.

2.2.2 Conservation of mass equation

attached text file: `output_d1q3_nse_mrt1_symbolic_pde_00.txt`

$\frac{\partial \rho}{\partial t} + \frac{v_1 \delta_l}{\delta_t} \frac{\partial \rho}{\partial x_1} + \frac{\rho \delta_l}{\delta_t} \frac{\partial v_1}{\partial x_1} + (-1 + v_1^2 + 3c_s^2) \frac{v_1 \delta_l^3}{12\delta_t} \frac{\partial^3 \rho}{\partial x_1^3} + (-1 + 3v_1^2 + c_s^2) \frac{\rho \delta_l^3}{12\delta_t} \frac{\partial^3 v_1}{\partial x_1^3} +$

$(-\omega_3 c_s^4 - 6v_1^2 - 12\omega_3 v_1^2 c_s^2 + 3\omega_3 v_1^2 + 2c_s^4 - 2c_s^2 - 3\omega_3 v_1^4 + 24v_1^2 c_s^2 + 6v_1^4 + \omega_3 c_s^2) \frac{\delta_l^4}{24\omega_3 \delta_t} \frac{\partial^4 \rho}{\partial x_1^4} +$

$(-4 + 10v_1^2 + 2\omega_3 - 5\omega_3 v_1^2 + 6c_s^2 - 3\omega_3 c_s^2) \frac{\rho v_1 \delta_l^4}{12\omega_3 \delta_t} \frac{\partial^4 v_1}{\partial x_1^4} = 0.$

2.2.3 Conservation of momentum equation

attached text file: `output_d1q3_nse_mrt1_symbolic_pde_01.txt`

$v_1 \frac{\partial \rho}{\partial t} + \rho \frac{\partial v_1}{\partial t} + (v_1^2 + c_s^2) \frac{\delta_l}{\delta_t} \frac{\partial \rho}{\partial x_1} + \frac{2\rho v_1 \delta_l}{\delta_t} \frac{\partial v_1}{\partial x_1} + (-2 + 6v_1^2 + \omega_3 - 3\omega_3 v_1^2 + 4c_s^2 - 2\omega_3 c_s^2) \frac{\delta_l^2}{\omega_3 \delta_t} \frac{\partial \rho}{\partial x_1} \frac{\partial v_1}{\partial x_1} +$

$(2 - \omega_3) \frac{3\rho v_1 \delta_l^2}{\omega_3 \delta_t} \left(\frac{\partial v_1}{\partial x_1}\right)^2 + (-2 + 2v_1^2 + \omega_3 - \omega_3 v_1^2 + 6c_s^2 - 3\omega_3 c_s^2) \frac{v_1 \delta_l^2}{2\omega_3 \delta_t} \frac{\partial^2 \rho}{\partial x_1^2} +$

$(-2 + 6v_1^2 + \omega_3 - 3\omega_3 v_1^2 + 2c_s^2 - \omega_3 c_s^2) \frac{\rho \delta_l^2}{2\omega_3 \delta_t} \frac{\partial^2 v_1}{\partial x_1^2} + \; C_1 \frac{\delta_l^3}{12\omega_3^2 \delta_t} \frac{\partial^3 \rho}{\partial x_1^3} +$

$(-24 + 60v_1^2 + 24\omega_3 + 5\omega_3^2 c_s^2 - 60\omega_3 v_1^2 - 4\omega_3^2 + 36c_s^2 + 11\omega_3^2 v_1^2 - 36\omega_3 c_s^2) \frac{\rho v_1 \delta_l^3}{6\omega_3^2 \delta_t} \frac{\partial^3 v_1}{\partial x_1^3} + \; C_2 \frac{v_1 \delta_l^4}{12\omega_3^3 \delta_t} \frac{\partial^4 \rho}{\partial x_1^4} +$

$C_3 \frac{\rho \delta_l^4}{12\omega_3^3 \delta_t} \frac{\partial^4 v_1}{\partial x_1^4} = 0,$

where:

$C_1 = -12\omega_3 c_s^4 - 36v_1^2 - 144\omega_3 v_1^2 c_s^2 - \omega_3^2 c_s^2 + 36\omega_3 v_1^2 + 12c_s^4 + 7\omega_3^2 v_1^4 - 12c_s^2 - 36\omega_3 v_1^4 - 7\omega_3^2 v_1^2 + 144v_1^2 c_s^2 + 36v_1^4 + 24\omega_3^2 v_1^2 c_s^2 + 12\omega_3 c_s^2 + \omega_3^2 c_s^4$

$C_2 = 12 - 216\omega_3 c_s^4 - 156v_1^2 - 18\omega_3 - 1008\omega_3 v_1^2 c_s^2 - 9\omega_3^3 v_1^4 - 78\omega_3^2 c_s^2 + 234\omega_3 v_1^2 + 144c_s^4 + 8\omega_3^2 - \omega_3^3 + 6\omega_3^3 c_s^2 - 34\omega_3^3 v_1^2 c_s^2 + 90\omega_3^2 v_1^4 - 132c_s^2 - 216\omega_3 v_1^4 - 98\omega_3^2 v_1^2 - 5\omega_3^3 c_s^4 + 672v_1^2 c_s^2 + 144v_1^4 + 404\omega_3^2 v_1^2 c_s^2 + 198\omega_3 c_s^2 + 82\omega_3^2 c_s^4 + 10\omega_3^3 v_1^2$

$C_3 = 12 - 36\omega_3 c_s^4 - 252v_1^2 - 18\omega_3 - 648\omega_3 v_1^2 c_s^2 - 29\omega_3^3 v_1^4 - 22\omega_3^2 c_s^2 + 378\omega_3 v_1^2 + 24c_s^4 + 8\omega_3^2 - \omega_3^3 + 2\omega_3^3 c_s^2 - 18\omega_3^3 v_1^2 c_s^2 + 310\omega_3^2 v_1^4 - 36c_s^2 - 756\omega_3 v_1^4 - 154\omega_3^2 v_1^2 - \omega_3^3 c_s^4 + 432v_1^2 c_s^2 + 504v_1^4 + 252\omega_3^2 v_1^2 c_s^2 + 54\omega_3 c_s^2 + 14\omega_3^2 c_s^4 + 14\omega_3^3 v_1^2$

2.3 CLBM

2.3.1 Definitions

Collision operator $\boldsymbol{C}$:

$$\boldsymbol{C}\left(\boldsymbol{f}\right) = \mathbf{K}^{-1}\mathbf{S}\left(\boldsymbol{\kappa}^{(eq)} - \mathbf{K}\boldsymbol{f}\right),$$

where

$$\mathbf{S} = \mathrm{diag}(\omega_1, \omega_2, \omega_3),$$

$\omega_1, \omega_2, \omega_3 \in (0, 2)$.
Matrix $\mathbf{K}$ corresponds to the transformation matrix to the central moment basis defined by

$$\boldsymbol{\kappa} = \left(k_{(0)}, k_{(1)}, k_{(2)}\right)^T$$

and is given by

$$\mathbf{K} = \begin{pmatrix} 1 & 1 & 1 \\ -v_1 & 1 - v_1 & -v_1 - 1 \\ v_1^2 & (1 - v_1)^2 & (v_1 + 1)^2 \end{pmatrix}.$$

The equilibrium central moments are defined by

$$\boldsymbol{\kappa}^{(eq)} = \mathbf{K}\mathbf{M}^{-1}\boldsymbol{\mu}^{(eq)},$$

i.e.,

$$\boldsymbol{\kappa}^{(eq)} = \left(\rho, 0, \rho c_s^2\right)^T.$$

2.3.2 Conservation of mass equation

attached text file: `output_d1q3_nse_clbm1_symbolic_pde_00.txt`

$\frac{\partial \rho}{\partial t} + \frac{v_1 \delta_l}{\delta_t}\frac{\partial \rho}{\partial x_1} + \frac{\rho \delta_l}{\delta_t}\frac{\partial v_1}{\partial x_1} + (-1 + v_1^2 + 3c_s^2)\frac{v_1 \delta_l^3}{12\delta_t}\frac{\partial^3 \rho}{\partial x_1^3} + (-1 + 3v_1^2 + c_s^2)\frac{\rho \delta_l^3}{12\delta_t}\frac{\partial^3 v_1}{\partial x_1^3} +$
$(2c_s^4 - 12v_1^2 c_s^2\omega_3 + 6v_1^4 - c_s^4\omega_3 + 3v_1^2\omega_3 - 6v_1^2 + c_s^2\omega_3 - 2c_s^2 - 3v_1^4\omega_3 + 24v_1^2 c_s^2)\frac{\delta_l^4}{24\omega_3\delta_t}\frac{\partial^4 \rho}{\partial x_1^4} +$
$(-4 - 5v_1^2\omega_3 + 10v_1^2 - 3c_s^2\omega_3 + 6c_s^2 + 2\omega_3)\frac{v_1 \rho \delta_l^4}{12\omega_3\delta_t}\frac{\partial^4 v_1}{\partial x_1^4} = 0.$

2.3.3 Conservation of momentum equation

attached text file: `output_d1q3_nse_clbm1_symbolic_pde_01.txt`

$$v_1\frac{\partial\rho}{\partial t} + \rho\frac{\partial v_1}{\partial t} + (v_1^2+c_s^2)\frac{\delta_l}{\delta_t}\frac{\partial\rho}{\partial x_1} + \frac{2v_1\rho\delta_l}{\delta_t}\frac{\partial v_1}{\partial x_1} + (-2-3v_1^2\omega_3+6v_1^2-2c_s^2\omega_3+4c_s^2+\omega_3)\frac{\delta_l^2}{\omega_3\delta_t}\frac{\partial\rho}{\partial x_1}\frac{\partial v_1}{\partial x_1} +$$

$$(2-\omega_3)\frac{3v_1\rho\delta_l^2}{\omega_3\delta_t}\left(\frac{\partial v_1}{\partial x_1}\right)^2 + (-2-v_1^2\omega_3+2v_1^2-3c_s^2\omega_3+6c_s^2+\omega_3)\frac{v_1\delta_l^2}{2\omega_3\delta_t}\frac{\partial^2\rho}{\partial x_1^2} +$$

$$(-2-3v_1^2\omega_3+6v_1^2-c_s^2\omega_3+2c_s^2+\omega_3)\frac{\rho\delta_l^2}{2\omega_3\delta_t}\frac{\partial^2 v_1}{\partial x_1^2} + C_1\frac{\delta_l^3}{12\omega_3^2\delta_t}\frac{\partial^3\rho}{\partial x_1^3} +$$

$$(-24+11v_1^2\omega_3^2-4\omega_3^2-60v_1^2\omega_3+60v_1^2+5c_s^2\omega_3^2-36c_s^2\omega_3+36c_s^2+24\omega_3)\frac{v_1\rho\delta_l^3}{6\omega_3^2\delta_t}\frac{\partial^3 v_1}{\partial x_1^3} + C_2\frac{v_1\delta_l^4}{12\omega_3^3\delta_t}\frac{\partial^4\rho}{\partial x_1^4} +$$

$$C_3\frac{\rho\delta_l^4}{12\omega_3^3\delta_t}\frac{\partial^4 v_1}{\partial x_1^4} = 0,$$

where:

$$C_1 = 12c_s^4 - 7v_1^2\omega_3^2 + c_s^4\omega_3^2 + 24v_1^2c_s^2\omega_3^2 - 144v_1^2c_s^2\omega_3 + 36v_1^4 - 12c_s^4\omega_3 + 36v_1^2\omega_3 - 36v_1^2 + 7v_1^4\omega_3^2 - c_s^2\omega_3^2 + 12c_s^2\omega_3 - 12c_s^2 - 36v_1^4\omega_3 + 144v_1^2c_s^2$$

$$C_2 = 12 + 144c_s^4 - 98v_1^2\omega_3^2 - \omega_3^3 + 82c_s^4\omega_3^2 - 34v_1^2c_s^2\omega_3^3 + 10v_1^2\omega_3^3 - 5c_s^4\omega_3^3 + 404v_1^2c_s^2\omega_3^2 + 8\omega_3^2 - 1008v_1^2c_s^2\omega_3 + 144v_1^4 - 216c_s^4\omega_3 + 234v_1^2\omega_3 - 9v_1^4\omega_3^3 - 156v_1^2 + 6c_s^2\omega_3^3 + 90v_1^4\omega_3^2 - 78c_s^2\omega_3^2 + 198c_s^2\omega_3 - 132c_s^2 - 18\omega_3 - 216v_1^4\omega_3 + 672v_1^2c_s^2$$

$$C_3 = 12 + 24c_s^4 - 154v_1^2\omega_3^2 - \omega_3^3 + 14c_s^4\omega_3^2 - 18v_1^2c_s^2\omega_3^3 + 14v_1^4\omega_3^3 - c_s^4\omega_3^3 + 252v_1^2c_s^2\omega_3^2 + 8\omega_3^2 - 648v_1^2c_s^2\omega_3 + 504v_1^4 - 36c_s^4\omega_3 + 378v_1^2\omega_3 - 29v_1^4\omega_3^3 - 252v_1^2 + 2c_s^2\omega_3^3 + 310v_1^4\omega_3^2 - 22c_s^2\omega_3^2 + 54c_s^2\omega_3 - 36c_s^2 - 18\omega_3 - 756v_1^4\omega_3 + 432v_1^2c_s^2$$

3 Comparison of SRT, MRT, and CLBM

3.1 Conservation of mass equation

$$\frac{\partial\rho}{\partial t} + v_1\frac{\delta_l}{\delta_t}\frac{\partial\rho}{\partial x_1} + \rho\frac{\delta_l}{\delta_t}\frac{\partial v_1}{\partial x_1} + (-1+v_1^2+3c_s^2)\frac{v_1}{12}\frac{\delta_l^3}{\delta_t}\frac{\partial^3\rho}{\partial x_1^3} + (-1+3v_1^2+c_s^2)\frac{\rho}{12}\frac{\delta_l^3}{\delta_t}\frac{\partial^3 v_1}{\partial x_1^3} + C_{\mathrm{D}_x^4\rho}^{(0)}\frac{\delta_l^4}{\delta_t}\frac{\partial^4\rho}{\partial x_1^4} + C_{\mathrm{D}_x^4 v_1}^{(0)}\frac{\delta_l^4}{\delta_t}\frac{\partial^4 v_1}{\partial x_1^4}$$
$$= 0,$$

where:

coefficient $C_{\mathrm{D}_x^4\rho}^{(0)}$ at $\frac{\partial^4\rho}{\partial x_1^4}$:

$$C_{\mathrm{D}_x^4\rho}^{(0),\mathrm{SRT}} = (c_s^2\omega - 3v_1^4\omega + 2c_s^4 - 6v_1^2 + 24v_1^2c_s^2 + 3v_1^2\omega + 6v_1^4 - 2c_s^2 - 12v_1^2c_s^2\omega - c_s^4\omega)\frac{1}{24\omega}$$

$$C_{\mathrm{D}_x^4\rho}^{(0),\mathrm{MRT1}} = (-3v_1^4\omega_3 + 2c_s^4 - 6v_1^2 + 24v_1^2c_s^2 + \omega_3c_s^2 + 6v_1^4 - 2c_s^2 - \omega_3c_s^4 + 3v_1^2\omega_3 - 12v_1^2\omega_3c_s^2)\frac{1}{24\omega_3}$$

$$C_{\mathrm{D}_x^4\rho}^{(0),\mathrm{CLBM1}} = C_{\mathrm{D}_x^4\rho}^{(0),\mathrm{MRT1}}$$

coefficient $C_{\mathrm{D}_x^4 v_1}^{(0)}$ at $\frac{\partial^4 v_1}{\partial x_1^4}$:

$$C_{\mathrm{D}_x^4 v_1}^{(0),\mathrm{SRT}} = (-4 - 3c_s^2\omega + 10v_1^2 - 5v_1^2\omega + 6c_s^2 + 2\omega)\frac{v_1\rho}{12\omega}$$

$$C_{\mathrm{D}_x^4 v_1}^{(0),\mathrm{MRT1}} = (-4 + 2\omega_3 + 10v_1^2 - 3\omega_3c_s^2 + 6c_s^2 - 5v_1^2\omega_3)\frac{v_1\rho}{12\omega_3}$$

$$C_{\mathrm{D}_x^4 v_1}^{(0),\mathrm{CLBM1}} = C_{\mathrm{D}_x^4 v_1}^{(0),\mathrm{MRT1}}$$

3.2 Conservation of momentum equation

$$v_1\frac{\partial\rho}{\partial t} + \rho\frac{\partial v_1}{\partial t} + (v_1^2+c_s^2)\frac{\delta_l}{\delta_t}\frac{\partial\rho}{\partial x_1} + 2v_1\rho\frac{\delta_l}{\delta_t}\frac{\partial v_1}{\partial x_1} + C_{\mathrm{D}_x\rho,\mathrm{D}_x v_1}^{(1)}\frac{\delta_l^2}{\delta_t}\frac{\partial\rho}{\partial x_1}\frac{\partial v_1}{\partial x_1} + C_{\mathrm{D}_x v_1,\mathrm{D}_x v_1}^{(1)}\frac{\delta_l^2}{\delta_t}\left(\frac{\partial v_1}{\partial x_1}\right)^2 + C_{\mathrm{D}_x^2\rho}^{(1)}\frac{\delta_l^2}{\delta_t}\frac{\partial^2\rho}{\partial x_1^2} +$$

$$C_{\mathrm{D}_x^2 v_1}^{(1)}\frac{\delta_l^2}{\delta_t}\frac{\partial^2 v_1}{\partial x_1^2} + C_{\mathrm{D}_x^3\rho}^{(1)}\frac{\delta_l^3}{\delta_t}\frac{\partial^3\rho}{\partial x_1^3} + C_{\mathrm{D}_x^3 v_1}^{(1)}\frac{\delta_l^3}{\delta_t}\frac{\partial^3 v_1}{\partial x_1^3} + C_{\mathrm{D}_x^4\rho}^{(1)}\frac{\delta_l^4}{\delta_t}\frac{\partial^4\rho}{\partial x_1^4} + C_{\mathrm{D}_x^4 v_1}^{(1)}\frac{\delta_l^4}{\delta_t}\frac{\partial^4 v_1}{\partial x_1^4} = 0,$$

where:

coefficient $C^{(1)}_{\mathrm{D}_x\rho,\mathrm{D}_x v_1}$ **at** $\frac{\partial\rho}{\partial x_1}\frac{\partial v_1}{\partial x_1}$**:**

$$C^{(1),\mathrm{SRT}}_{\mathrm{D}_x\rho,\mathrm{D}_x v_1} = (-2 - 2c_s^2\omega + 6v_1^2 - 3v_1^2\omega + 4c_s^2 + \omega)\tfrac{1}{\omega}$$

$$C^{(1),\mathrm{MRT1}}_{\mathrm{D}_x\rho,\mathrm{D}_x v_1} = (-2 + \omega_3 + 6v_1^2 - 2\omega_3 c_s^2 + 4c_s^2 - 3v_1^2\omega_3)\tfrac{1}{\omega_3}$$

$$C^{(1),\mathrm{CLBM1}}_{\mathrm{D}_x\rho,\mathrm{D}_x v_1} = C^{(1),\mathrm{MRT1}}_{\mathrm{D}_x\rho,\mathrm{D}_x v_1}$$

coefficient $C^{(1)}_{\mathrm{D}_x v_1,\mathrm{D}_x v_1}$ **at** $\left(\frac{\partial v_1}{\partial x_1}\right)^2$**:**

$$C^{(1),\mathrm{SRT}}_{\mathrm{D}_x v_1,\mathrm{D}_x v_1} = (2 - \omega)\tfrac{3v_1\rho}{\omega}$$

$$C^{(1),\mathrm{MRT1}}_{\mathrm{D}_x v_1,\mathrm{D}_x v_1} = (2 - \omega_3)\tfrac{3v_1\rho}{\omega_3}$$

$$C^{(1),\mathrm{CLBM1}}_{\mathrm{D}_x v_1,\mathrm{D}_x v_1} = C^{(1),\mathrm{MRT1}}_{\mathrm{D}_x v_1,\mathrm{D}_x v_1}$$

coefficient $C^{(1)}_{\mathrm{D}_x^2\rho}$ **at** $\frac{\partial^2\rho}{\partial x_1^2}$**:**

$$C^{(1),\mathrm{SRT}}_{\mathrm{D}_x^2\rho} = (-2 - 3c_s^2\omega + 2v_1^2 - v_1^2\omega + 6c_s^2 + \omega)\tfrac{v_1}{2\omega}$$

$$C^{(1),\mathrm{MRT1}}_{\mathrm{D}_x^2\rho} = (-2 + \omega_3 + 2v_1^2 - 3\omega_3 c_s^2 + 6c_s^2 - v_1^2\omega_3)\tfrac{v_1}{2\omega_3}$$

$$C^{(1),\mathrm{CLBM1}}_{\mathrm{D}_x^2\rho} = C^{(1),\mathrm{MRT1}}_{\mathrm{D}_x^2\rho}$$

coefficient $C^{(1)}_{\mathrm{D}_x^2 v_1}$ **at** $\frac{\partial^2 v_1}{\partial x_1^2}$**:**

$$C^{(1),\mathrm{SRT}}_{\mathrm{D}_x^2 v_1} = (-2 - c_s^2\omega + 6v_1^2 - 3v_1^2\omega + 2c_s^2 + \omega)\tfrac{\rho}{2\omega}$$

$$C^{(1),\mathrm{MRT1}}_{\mathrm{D}_x^2 v_1} = (-2 + \omega_3 + 6v_1^2 - \omega_3 c_s^2 + 2c_s^2 - 3v_1^2\omega_3)\tfrac{\rho}{2\omega_3}$$

$$C^{(1),\mathrm{CLBM1}}_{\mathrm{D}_x^2 v_1} = C^{(1),\mathrm{MRT1}}_{\mathrm{D}_x^2 v_1}$$

coefficient $C^{(1)}_{\mathrm{D}_x^3\rho}$ **at** $\frac{\partial^3\rho}{\partial x_1^3}$**:**

$$C^{(1),\mathrm{SRT}}_{\mathrm{D}_x^3\rho} = (12c_s^2\omega - 36v_1^4\omega + 12c_s^4 + 7v_1^4\omega^2 - 36v_1^2 + 144v_1^2 c_s^2 - c_s^2\omega^2 + 36v_1^2\omega + 36v_1^4 - 12c_s^2 - 144v_1^2 c_s^2\omega - 12c_s^4\omega + c_s^4\omega^2 + 24v_1^2 c_s^2\omega^2 - 7v_1^2\omega^2)\tfrac{1}{12\omega^2}$$

$$C^{(1),\mathrm{MRT1}}_{\mathrm{D}_x^3\rho} = (-36v_1^4\omega_3 + 24v_1^2\omega_3^2 c_s^2 + \omega_3^2 c_s^4 + 12c_s^4 - 36v_1^2 + 144v_1^2 c_s^2 + 12\omega_3 c_s^2 + 7v_1^4\omega_3^2 + 36v_1^4 - \omega_3^2 c_s^2 - 12c_s^2 - 12\omega_3 c_s^4 + 36v_1^2\omega_3 - 7v_1^2\omega_3^2 - 144v_1^2\omega_3 c_s^2)\tfrac{1}{12\omega_3^2}$$

$$C^{(1),\mathrm{CLBM1}}_{\mathrm{D}_x^3\rho} = C^{(1),\mathrm{MRT1}}_{\mathrm{D}_x^3\rho}$$

coefficient $C^{(1)}_{\mathrm{D}_x^3 v_1}$ **at** $\frac{\partial^3 v_1}{\partial x_1^3}$**:**

$$C^{(1),\mathrm{SRT}}_{\mathrm{D}_x^3 v_1} = (-24 - 4\omega^2 - 36c_s^2\omega + 60v_1^2 + 5c_s^2\omega^2 - 60v_1^2\omega + 36c_s^2 + 24\omega + 11v_1^2\omega^2)\tfrac{v_1\rho}{6\omega^2}$$

$$C^{(1),\mathrm{MRT1}}_{\mathrm{D}_x^3 v_1} = (-24 - 4\omega_3^2 + 24\omega_3 + 60v_1^2 - 36\omega_3 c_s^2 + 5\omega_3^2 c_s^2 + 36c_s^2 - 60v_1^2\omega_3 + 11v_1^2\omega_3^2)\tfrac{v_1\rho}{6\omega_3^2}$$

$$C^{(1),\mathrm{CLBM1}}_{\mathrm{D}_x^3 v_1} = C^{(1),\mathrm{MRT1}}_{\mathrm{D}_x^3 v_1}$$

coefficient $C^{(1)}_{\mathrm{D}_x^4\rho}$ **at** $\frac{\partial^4\rho}{\partial x_1^4}$**:**

$C_{\mathrm{D}_x^4 \rho}^{(1),\mathrm{SRT}} = (12 + 8\omega^2 + 198c_s^2\omega - 216v_1^4\omega - \omega^3 + 144c_s^4 + 6c_s^2\omega^3 + 90v_1^4\omega^2 - 156v_1^2 + 672v_1^2c_s^2 - 9v_1^4\omega^3 - 78c_s^2\omega^2 + 234v_1^2\omega + 144v_1^4 - 132c_s^2 - 1008v_1^2c_s^2\omega - 216c_s^4\omega - 18\omega - 34v_1^2c_s^2\omega^3 + 10v_1^2\omega^3 + 82c_s^4\omega^2 + 404v_1^2c_s^2\omega^2 - 5c_s^4\omega^3 - 98v_1^2\omega^2)\frac{v_1}{12\omega^3}$

$C_{\mathrm{D}_x^4 \rho}^{(1),\mathrm{MRT1}} = (12 - 216v_1^4\omega_3 + 404v_1^2\omega_3^2c_s^2 - \omega_3^3 - 5\omega_3^3c_s^4 + 8\omega_3^2 - 18\omega_3 + 82\omega_3^2c_s^4 + 144c_s^4 - 9v_1^4\omega_3^3 - 156v_1^2 - 34v_1^2\omega_3^3c_s^2 + 672v_1^2c_s^2 + 198\omega_3c_s^2 + 90v_1^4\omega_3^2 + 144v_1^4 - 78\omega_3^2c_s^2 - 132c_s^2 - 216\omega_3c_s^4 + 234v_1^2\omega_3 + 6\omega_3^3c_s^2 - 98v_1^2\omega_3^2 - 1008v_1^2\omega_3c_s^2 + 10v_1^2\omega_3^3)\frac{v_1}{12\omega_3^3}$

$C_{\mathrm{D}_x^4 \rho}^{(1),\mathrm{CLBM1}} = C_{\mathrm{D}_x^4 \rho}^{(1),\mathrm{MRT1}}$

coefficient $C_{\mathrm{D}_x^4 v_1}^{(1)}$ **at** $\frac{\partial^4 v_1}{\partial x_1^4}$:

$C_{\mathrm{D}_x^4 v_1}^{(1),\mathrm{SRT}} = (12 + 8\omega^2 + 54c_s^2\omega - 756v_1^4\omega - \omega^3 + 24c_s^4 + 2c_s^2\omega^3 + 310v_1^4\omega^2 - 252v_1^2 + 432v_1^2c_s^2 - 29v_1^4\omega^3 - 22c_s^2\omega^2 + 378v_1^2\omega + 504v_1^4 - 36c_s^2 - 648v_1^2c_s^2\omega - 36c_s^4\omega - 18\omega - 18v_1^2c_s^2\omega^3 + 14v_1^2\omega^3 + 14c_s^4\omega^2 + 252v_1^2c_s^2\omega^2 - c_s^4\omega^3 - 154v_1^2\omega^2)\frac{\rho}{12\omega^3}$

$C_{\mathrm{D}_x^4 v_1}^{(1),\mathrm{MRT1}} = (12 - 756v_1^4\omega_3 + 252v_1^2\omega_3^2c_s^2 - \omega_3^3 - \omega_3^3c_s^4 + 8\omega_3^2 - 18\omega_3 + 14\omega_3^2c_s^4 + 24c_s^4 - 29v_1^4\omega_3^3 - 252v_1^2 - 18v_1^2\omega_3^3c_s^2 + 432v_1^2c_s^2 + 54\omega_3c_s^2 + 310v_1^4\omega_3^2 + 504v_1^4 - 22\omega_3^2c_s^2 - 36c_s^2 - 36\omega_3c_s^4 + 378v_1^2\omega_3 + 2\omega_3^3c_s^2 - 154v_1^2\omega_3^2 - 648v_1^2\omega_3c_s^2 + 14v_1^2\omega_3^3)\frac{\rho}{12\omega_3^3}$

$C_{\mathrm{D}_x^4 v_1}^{(1),\mathrm{CLBM1}} = C_{\mathrm{D}_x^4 v_1}^{(1),\mathrm{MRT1}}$

References

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