

D1Q3 NSE,

a supplementary material for

Lattice Boltzmann Method Analysis Tool (LBMAT)

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1 Global definitions

In \mathbb{R}^1 , the position and velocity vectors are given by $\mathbf{x} = (x_1)$ and $\mathbf{v} = (v_1)$, respectively.

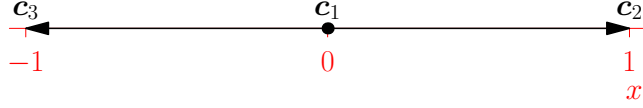
1.1 Discrete velocity vectors

Discrete velocity vectors and the lattice speed of sound are defined by

$$\{\mathbf{c}_i\}_{i=1}^3 = ((0), (1), (-1)),$$

$$c_s = \frac{1}{\sqrt{3}},$$

respectively [1].



1.2 Raw and central moments

The raw and central moments are defined by

$$m_{\boldsymbol{\alpha}} := \sum_{i=1}^3 f_i \mathbf{c}_i^{\boldsymbol{\alpha}},$$

and

$$k_{\boldsymbol{\alpha}} := \sum_{i=1}^3 f_i (\mathbf{c}_i - \mathbf{v})^{\boldsymbol{\alpha}},$$

respectively, where $\boldsymbol{\alpha} = (\alpha_1) \in \mathbb{Z}^1$ denotes a multi-index and $\mathbf{c}_i^{\boldsymbol{\alpha}} := [\mathbf{c}_i]_1^{\alpha_1}$.

1.3 Transformation matrix M

Matrix \mathbf{M} , that defines macroscopic quantities (moments) $\boldsymbol{\mu}$ by

$$\boldsymbol{\mu} = \mathbf{M}\mathbf{f},$$

with $\mathbf{f} = (f_1, f_2, f_3)^T$, is selected such that

$$\boldsymbol{\mu} = \left(m_{(0)}, m_{(1)}, m_{(2)} \right)^T,$$

i.e., \mathbf{M} is given by

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

1.4 Equilibrium

The corresponding equilibrium raw moments are defined using the continuous Maxwell–Boltzmann distribution function [1]

$$f^{(eq)}(\xi) = \frac{\rho}{(2\pi c_s^2)^{\frac{1}{2}}} \exp\left(-\frac{(\xi - v_1)^2}{2c_s^2}\right)$$

as

$$m_{(\alpha)}^{(eq)} = \int_{\mathbb{R}} \xi^\alpha f^{(eq)}(\xi) d\xi,$$

where $\alpha \in \{0, 1, 2\}$. Hence, the equilibrium moments $\boldsymbol{\mu}^{(eq)}$ satisfy

$$\boldsymbol{\mu}^{(eq)} = \left(\rho, \rho v_1, \rho(v_1^2 + c_s^2) \right)^T.$$

2 Spatial EPDEs

2.1 SRT

2.1.1 Definitions

Collision operator \mathbf{C} :

$$\mathbf{C}(\mathbf{f}) = \omega \left(\mathbf{M}^{-1} \boldsymbol{\mu}^{(eq)} - \mathbf{f} \right),$$

$\omega \in (0, 2)$.

2.1.2 Conservation of mass equation



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$$\begin{aligned} & \frac{\partial \rho}{\partial t} + \frac{v_1 \delta_l}{\delta_t} \frac{\partial \rho}{\partial x_1} + \frac{\rho \delta_l}{\delta_t} \frac{\partial v_1}{\partial x_1} + (-1 + v_1^2 + 3cs^2) \frac{v_1 \delta_l^3}{12\delta_t} \frac{\partial^3 \rho}{\partial x_1^3} + (-1 + 3v_1^2 + cs^2) \frac{\rho \delta_l^3}{12\delta_t} \frac{\partial^3 v_1}{\partial x_1^3} + \\ & (3\omega v_1^2 - 6v_1^2 + 24cs^2 v_1^2 - cs^4 \omega + 2cs^4 - 12cs^2 \omega v_1^2 - 2cs^2 + 6v_1^4 - 3\omega v_1^4 + cs^2 \omega) \frac{\delta_l^4}{24\omega \delta_t} \frac{\partial^4 \rho}{\partial x_1^4} + \\ & (-4 - 5\omega v_1^2 + 10v_1^2 + 2\omega + 6cs^2 - 3cs^2 \omega) \frac{\rho v_1 \delta_l^4}{12\omega \delta_t} \frac{\partial^4 v_1}{\partial x_1^4} = 0. \end{aligned}$$

2.1.3 Conservation of momentum equation



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$$\begin{aligned}
& v_1 \frac{\partial \rho}{\partial t} + \rho \frac{\partial v_1}{\partial t} + (v_1^2 + cs^2) \frac{\delta_l}{\delta_t} \frac{\partial \rho}{\partial x_1} + \frac{2\rho v_1 \delta_l}{\delta_t} \frac{\partial v_1}{\partial x_1} + (-2 - 3\omega v_1^2 + 6v_1^2 + \omega + 4cs^2 - 2\omega cs^2) \frac{\delta_l^2}{\omega \delta_t} \frac{\partial \rho}{\partial x_1} \frac{\partial v_1}{\partial x_1} + \\
& (2 - \omega) \frac{3\rho v_1 \delta_l^2}{\omega \delta_t} \left(\frac{\partial v_1}{\partial x_1} \right)^2 + (-2 - \omega v_1^2 + 2v_1^2 + \omega + 6cs^2 - 3\omega cs^2) \frac{v_1 \delta_l^2}{2\omega \delta_t} \frac{\partial^2 \rho}{\partial x_1^2} + \\
& (-2 - 3\omega v_1^2 + 6v_1^2 + \omega + 2cs^2 - \omega cs^2) \frac{\rho \delta_l^2}{2\omega \delta_t} \frac{\partial^2 v_1}{\partial x_1^2} + C_1 \frac{\delta_l^3}{12\omega^2 \delta_t} \frac{\partial^3 \rho}{\partial x_1^3} + \\
& (-24 - 60\omega v_1^2 + 60v_1^2 + 24\omega + 36cs^2 - 36\omega cs^2 + 5\omega^2 cs^2 + 11\omega^2 v_1^2 - 4\omega^2) \frac{\rho v_1 \delta_l^3}{6\omega^2 \delta_t} \frac{\partial^3 v_1}{\partial x_1^3} + C_2 \frac{v_1 \delta_l^4}{12\omega^3 \delta_t} \frac{\partial^4 \rho}{\partial x_1^4} + \\
& C_3 \frac{\rho \delta_l^4}{12\omega^3 \delta_t} \frac{\partial^4 v_1}{\partial x_1^4} = 0,
\end{aligned}$$

where:

$$C_1 = 36\omega v_1^2 - 36v_1^2 + 7\omega^2 v_1^4 - 12cs^2 + 24\omega^2 v_1^2 cs^2 + 12\omega cs^2 + 144v_1^2 cs^2 + \omega^2 cs^4 - 12\omega cs^4 - 144\omega v_1^2 cs^2 + 12cs^4 - \omega^2 cs^2 + 36v_1^4 - 36\omega v_1^4 - 7\omega^2 v_1^2$$

$$C_2 = 12 + 234\omega v_1^2 - 156v_1^2 + 90\omega^2 v_1^4 - 5\omega^3 cs^4 - 18\omega - 132cs^2 + 404\omega^2 v_1^2 cs^2 + 198\omega cs^2 + 672v_1^2 cs^2 + 82\omega^2 cs^4 - 9\omega^3 v_1^4 - 216\omega cs^4 - 1008\omega v_1^2 cs^2 + 144cs^4 - 78\omega^2 cs^2 + 10\omega^3 v_1^2 + 144v_1^4 - 216\omega v_1^4 - \omega^3 - 34\omega^3 v_1^2 cs^2 - 98\omega^2 v_1^2 + 6\omega^3 cs^2 + 8\omega^2$$

$$C_3 = 12 + 378\omega v_1^2 - 252v_1^2 + 310\omega^2 v_1^4 - \omega^3 cs^4 - 18\omega - 36cs^2 + 252\omega^2 v_1^2 cs^2 + 54\omega cs^2 + 432v_1^2 cs^2 + 14\omega^2 cs^4 - 29\omega^3 v_1^4 - 36\omega cs^4 - 648\omega v_1^2 cs^2 + 24cs^4 - 22\omega^2 cs^2 + 14\omega^3 v_1^2 + 504v_1^4 - 756\omega v_1^4 - \omega^3 - 18\omega^3 v_1^2 cs^2 - 154\omega^2 v_1^2 + 2\omega^3 cs^2 + 8\omega^2$$

2.2 MRT

2.2.1 Definitions

Collision operator C :

$$C(f) = \mathbf{M}^{-1} \mathbf{S} \left(\mu^{(eq)} - \mathbf{M} f \right),$$

where

$$\mathbf{S} = \text{diag}(\omega_1, \omega_2, \omega_3),$$

$$\omega_1, \omega_2, \omega_3 \in (0, 2).$$

2.2.2 Conservation of mass equation



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$$\begin{aligned}
& \frac{\partial \rho}{\partial t} + \frac{v_1 \delta_l}{\delta_t} \frac{\partial \rho}{\partial x_1} + \frac{\rho \delta_l}{\delta_t} \frac{\partial v_1}{\partial x_1} + (-1 + v_1^2 + 3cs^2) \frac{v_1 \delta_l^3}{12\delta_t} \frac{\partial^3 \rho}{\partial x_1^3} + (-1 + 3v_1^2 + cs^2) \frac{\rho \delta_l^3}{12\delta_t} \frac{\partial^3 v_1}{\partial x_1^3} + \\
& (24v_1^2 cs^2 + 3\omega_3 v_1^2 - 6v_1^2 + \omega_3 cs^2 - 2cs^2 - 12\omega_3 v_1^2 cs^2 + 2cs^4 - \omega_3 cs^4 + 6v_1^4 - 3\omega_3 v_1^4) \frac{\delta_l^4}{24\omega_3 \delta_t} \frac{\partial^4 \rho}{\partial x_1^4} + \\
& (-4 - 5\omega_3 v_1^2 + 10v_1^2 + 2\omega_3 - 3\omega_3 cs^2 + 6cs^2) \frac{v_1 \rho \delta_l^4}{12\omega_3 \delta_t} \frac{\partial^4 v_1}{\partial x_1^4} = 0.
\end{aligned}$$

2.2.3 Conservation of momentum equation



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$$\begin{aligned}
& v_1 \frac{\partial \rho}{\partial t} + \rho \frac{\partial v_1}{\partial t} + (v_1^2 + cs^2) \frac{\delta_l}{\delta_t} \frac{\partial \rho}{\partial x_1} + \frac{2v_1 \rho \delta_l}{\delta_t} \frac{\partial v_1}{\partial x_1} + (-2 - 3\omega_3 v_1^2 + 6v_1^2 + \omega_3 - 2\omega_3 cs^2 + 4cs^2) \frac{\delta_l^2}{\omega_3 \delta_t} \frac{\partial \rho}{\partial x_1} \frac{\partial v_1}{\partial x_1} + \\
& (2 - \omega_3) \frac{3v_1 \rho \delta_l^2}{\omega_3 \delta_t} \left(\frac{\partial v_1}{\partial x_1} \right)^2 + (-2 - \omega_3 v_1^2 + 2v_1^2 + \omega_3 - 3\omega_3 cs^2 + 6cs^2) \frac{v_1 \delta_l^2}{2\omega_3 \delta_t} \frac{\partial^2 \rho}{\partial x_1^2} + \\
& (-2 - 3\omega_3 v_1^2 + 6v_1^2 + \omega_3 - \omega_3 cs^2 + 2cs^2) \frac{\rho \delta_l^2}{2\omega_3 \delta_t} \frac{\partial^2 v_1}{\partial x_1^2} + C_1 \frac{\delta_l^3}{12\omega_3^2 \delta_t} \frac{\partial^3 \rho}{\partial x_1^3} + \\
& (-24 + 5\omega_3^2 cs^2 - 60\omega_3 v_1^2 + 60v_1^2 + 24\omega_3 - 4\omega_3^2 + 11\omega_3^2 v_1^2 - 36\omega_3 cs^2 + 36cs^2) \frac{v_1 \rho \delta_l^3}{6\omega_3^2 \delta_t} \frac{\partial^3 v_1}{\partial x_1^3} + C_2 \frac{v_1 \delta_l^4}{12\omega_3^3 \delta_t} \frac{\partial^4 \rho}{\partial x_1^4} + \\
& C_3 \frac{\rho \delta_l^4}{12\omega_3^3 \delta_t} \frac{\partial^4 v_1}{\partial x_1^4} = 0,
\end{aligned}$$

where:

$$C_1 = -\omega_3^2 cs^2 + 7\omega_3^2 v_1^4 + 36\omega_3 v_1^2 + 12cs^4 - 36v_1^2 - 12\omega_3 cs^4 + 24\omega_3^2 cs^2 v_1^2 - 144\omega_3 cs^2 v_1^2 - 7\omega_3^2 v_1^2 + \omega_3^2 cs^4 + 144cs^2 v_1^2 + 36v_1^4 + 12\omega_3 cs^2 - 36\omega_3 v_1^4 - 12cs^2$$

$$C_2 = 12 - 78\omega_3^2 cs^2 + 90\omega_3^2 v_1^4 + 234\omega_3 v_1^2 + 144cs^4 - 156v_1^2 - 216\omega_3 cs^4 - 18\omega_3 - \omega_3^3 - 9\omega_3^3 v_1^4 + 6\omega_3^3 cs^2 + 8\omega_3^2 + 404\omega_3^2 cs^2 v_1^2 - 5\omega_3^3 cs^4 + 10\omega_3^3 v_1^2 - 1008\omega_3 cs^2 v_1^2 - 98\omega_3^2 v_1^2 - 34\omega_3^3 cs^2 v_1^2 + 82\omega_3^2 cs^4 + 672cs^2 v_1^2 + 144v_1^4 + 198\omega_3 cs^2 - 216\omega_3 v_1^4 - 132cs^2$$

$$C_3 = 12 - 22\omega_3^2 cs^2 + 310\omega_3^2 v_1^4 + 378\omega_3 v_1^2 + 24cs^4 - 252v_1^2 - 36\omega_3 cs^4 - 18\omega_3 - \omega_3^3 - 29\omega_3^3 v_1^4 + 2\omega_3^3 cs^2 + 8\omega_3^2 + 252\omega_3^2 cs^2 v_1^2 - \omega_3^3 cs^4 + 14\omega_3^3 v_1^2 - 648\omega_3 cs^2 v_1^2 - 154\omega_3^2 v_1^2 - 18\omega_3^3 cs^2 v_1^2 + 14\omega_3^2 cs^4 + 432cs^2 v_1^2 + 504v_1^4 + 54\omega_3 cs^2 - 756\omega_3 v_1^4 - 36cs^2$$

2.3 CLBM

2.3.1 Definitions

Collision operator C :

$$C(f) = \mathbf{K}^{-1} \mathbf{S} \left(\boldsymbol{\kappa}^{(eq)} - \mathbf{K} f \right),$$

where

$$\mathbf{S} = \text{diag}(\omega_1, \omega_2, \omega_3),$$

$$\omega_1, \omega_2, \omega_3 \in (0, 2).$$

Matrix \mathbf{K} corresponds to the transformation matrix to the central moment basis defined by

$$\boldsymbol{\kappa} = \left(k_{(0)}, k_{(1)}, k_{(2)} \right)^T$$

and is given by

$$\mathbf{K} = \begin{pmatrix} 1 & 1 & 1 \\ -v_1 & 1 - v_1 & -v_1 - 1 \\ v_1^2 & (1 - v_1)^2 & (v_1 + 1)^2 \end{pmatrix}.$$

The equilibrium central moments are defined by

$$\boldsymbol{\kappa}^{(eq)} = \mathbf{K} \mathbf{M}^{-1} \boldsymbol{\mu}^{(eq)},$$

i.e.,

$$\boldsymbol{\kappa}^{(eq)} = \left(\rho, 0, \rho c_s^2 \right)^T.$$

2.3.2 Conservation of mass equation



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$$\begin{aligned} & \frac{\partial \rho}{\partial t} + \frac{\delta_l v_1}{\delta_t} \frac{\partial \rho}{\partial x_1} + \frac{\delta_l \rho}{\delta_t} \frac{\partial v_1}{\partial x_1} + (-1 + v_1^2 + 3cs^2) \frac{\delta_l^3 v_1}{12\delta_t} \frac{\partial^3 \rho}{\partial x_1^3} + (-1 + 3v_1^2 + cs^2) \frac{\delta_l^3 \rho}{12\delta_t} \frac{\partial^3 v_1}{\partial x_1^3} + \\ & (3\omega_3 v_1^2 + 2cs^4 + 6v_1^4 + \omega_3 cs^2 - 6v_1^2 - \omega_3 cs^4 + 24v_1^2 cs^2 - 12\omega_3 v_1^2 cs^2 - 3\omega_3 v_1^4 - 2cs^2) \frac{\delta_l^4}{24\omega_3 \delta_t} \frac{\partial^4 \rho}{\partial x_1^4} + \\ & (-4 - 5\omega_3 v_1^2 - 3\omega_3 cs^2 + 2\omega_3 + 10v_1^2 + 6cs^2) \frac{\delta_l^4 v_1 \rho}{12\omega_3 \delta_t} \frac{\partial^4 v_1}{\partial x_1^4} = 0. \end{aligned}$$

2.3.3 Conservation of momentum equation



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$$\begin{aligned}
& v_1 \frac{\partial \rho}{\partial t} + \rho \frac{\partial v_1}{\partial t} + (v_1^2 + cs^2) \frac{\delta_l}{\delta_t} \frac{\partial \rho}{\partial x_1} + \frac{2\delta_l v_1 \rho}{\delta_t} \frac{\partial v_1}{\partial x_1} + (-2 - 3\omega_3 v_1^2 - 2\omega_3 cs^2 + \omega_3 + 6v_1^2 + 4cs^2) \frac{\delta_l^2}{\omega_3 \delta_t} \frac{\partial \rho}{\partial x_1} \frac{\partial v_1}{\partial x_1} + \\
& (2 - \omega_3) \frac{3\delta_l^2 v_1 \rho}{\omega_3 \delta_t} \left(\frac{\partial v_1}{\partial x_1} \right)^2 + (-2 - \omega_3 v_1^2 - 3\omega_3 cs^2 + \omega_3 + 2v_1^2 + 6cs^2) \frac{\delta_l^2 v_1}{2\omega_3 \delta_t} \frac{\partial^2 \rho}{\partial x_1^2} + \\
& (-2 - 3\omega_3 v_1^2 - \omega_3 cs^2 + \omega_3 + 6v_1^2 + 2cs^2) \frac{\delta_l^2 \rho}{2\omega_3 \delta_t} \frac{\partial^2 v_1}{\partial x_1^2} + C_1 \frac{\delta_l^3}{12\omega_3^2 \delta_t} \frac{\partial^3 \rho}{\partial x_1^3} + \\
& (-24 - 60\omega_3 v_1^2 - 36\omega_3 cs^2 - 4\omega_3^2 + 24\omega_3 + 60v_1^2 + 36cs^2 + 11\omega_3^2 v_1^2 + 5\omega_3^2 cs^2) \frac{\delta_l^3 v_1 \rho}{6\omega_3^2 \delta_t} \frac{\partial^3 v_1}{\partial x_1^3} + C_2 \frac{\delta_l^4 v_1}{12\omega_3^3 \delta_t} \frac{\partial^4 \rho}{\partial x_1^4} + \\
& C_3 \frac{\delta_l^4 \rho}{12\omega_3^3 \delta_t} \frac{\partial^4 v_1}{\partial x_1^4} = 0,
\end{aligned}$$

where:

$$\begin{aligned}
C_1 &= 7\omega_3^2 v_1^4 + \omega_3^2 cs^4 + 36\omega_3 v_1^2 + 12\omega_3 cs^2 + 36v_1^4 + 12cs^4 + 24\omega_3^2 cs^2 v_1^2 - 144\omega_3 cs^2 v_1^2 - 36v_1^2 - 12cs^2 + 144cs^2 v_1^2 - 7\omega_3^2 v_1^2 - \omega_3^2 cs^2 - 36\omega_3 v_1^4 - 12\omega_3 cs^4 \\
C_2 &= 12 - \omega_3^3 + 90\omega_3^2 v_1^4 + 82\omega_3^2 cs^4 + 234\omega_3 v_1^2 + 198\omega_3 cs^2 + 8\omega_3^2 - 5\omega_3^3 cs^4 + 144v_1^4 - 9\omega_3^3 v_1^4 + 144cs^4 + 404\omega_3^2 cs^2 v_1^2 - 18\omega_3 - 1008\omega_3 cs^2 v_1^2 + \\
& 6\omega_3^3 cs^2 - 156v_1^2 + 10\omega_3^3 v_1^2 - 132cs^2 + 672cs^2 v_1^2 - 98\omega_3^2 v_1^2 - 78\omega_3^2 cs^2 - 34\omega_3^3 cs^2 v_1^2 - 216\omega_3 v_1^4 - 216\omega_3 cs^4 \\
C_3 &= 12 - \omega_3^3 + 310\omega_3^2 v_1^4 + 14\omega_3^2 cs^4 + 378\omega_3 v_1^2 + 54\omega_3 cs^2 + 8\omega_3^2 - \omega_3^3 cs^4 + 504v_1^4 - 29\omega_3^3 v_1^4 + 24cs^4 + 252\omega_3^2 cs^2 v_1^2 - 18\omega_3 - 648\omega_3 cs^2 v_1^2 + \\
& 2\omega_3^3 cs^2 - 252v_1^2 + 14\omega_3^3 v_1^2 - 36cs^2 + 432cs^2 v_1^2 - 154\omega_3^2 v_1^2 - 22\omega_3^2 cs^2 - 18\omega_3^3 cs^2 v_1^2 - 756\omega_3 v_1^4 - 36\omega_3 cs^4
\end{aligned}$$

3 Comparison of SRT, MRT, and CLBM

3.1 Conservation of mass equation

$$\frac{\partial \rho}{\partial t} + v_1 \frac{\delta_l}{\delta_t} \frac{\partial \rho}{\partial x_1} + \rho \frac{\delta_l}{\delta_t} \frac{\partial v_1}{\partial x_1} + C_{D_x^3 \rho}^{(0)} \frac{\delta_l^3}{\delta_t} \frac{\partial^3 \rho}{\partial x_1^3} + C_{D_x^3 v_1}^{(0)} \frac{\delta_l^3}{\delta_t} \frac{\partial^3 v_1}{\partial x_1^3} + C_{D_x^4 \rho}^{(0)} \frac{\delta_l^4}{\delta_t} \frac{\partial^4 \rho}{\partial x_1^4} + C_{D_x^4 v_1}^{(0)} \frac{\delta_l^4}{\delta_t} \frac{\partial^4 v_1}{\partial x_1^4} = 0,$$

where:

coefficient $C_{D_x^3 \rho}^{(0)}$ at $\frac{\partial^3 \rho}{\partial x_1^3}$:

$$C_{D_x^3 \rho}^{(0), \text{SRT}} = (-1 + v_1^2 + 3cs^2) \frac{v_1}{12}$$

$$C_{D_x^3 \rho}^{(0), \text{MRT1}} = (-1 + v_1^2 + 3cs^2) \frac{v_1}{12}$$

$$C_{D_x^3 \rho}^{(0), \text{CLBM1}} = (-1 + 3cs^2 + v_1^2) \frac{v_1}{12}$$

coefficient $C_{D_x^3 v_1}^{(0)}$ at $\frac{\partial^3 v_1}{\partial x_1^3}$:

$$C_{D_x^3 v_1}^{(0), \text{SRT}} = (-1 + 3v_1^2 + cs^2) \frac{\rho}{12}$$

$$C_{D_x^3 v_1}^{(0), \text{MRT1}} = (-1 + 3v_1^2 + cs^2) \frac{\rho}{12}$$

$$C_{D_x^3 v_1}^{(0), \text{CLBM1}} = (-1 + cs^2 + 3v_1^2) \frac{\rho}{12}$$

coefficient $C_{D_x^4 \rho}^{(0)}$ at $\frac{\partial^4 \rho}{\partial x_1^4}$:

$$C_{D_x^4 \rho}^{(0), \text{SRT}} = (2cs^4 - 12v_1^2 \omega cs^2 + 24v_1^2 cs^2 - 6v_1^2 + \omega cs^2 + 3v_1^2 \omega + 6v_1^4 - \omega cs^4 - 2cs^2 - 3v_1^4 \omega) \frac{1}{24\omega}$$

$$C_{D_x^4 \rho}^{(0), \text{MRT1}} = (\omega_3 cs^2 - 12v_1^2 \omega_3 cs^2 + 2cs^4 + 3v_1^2 \omega_3 - 6v_1^2 + 24v_1^2 cs^2 + 6v_1^4 - \omega_3 cs^4 - 2cs^2 - 3v_1^4 \omega_3) \frac{1}{24\omega_3}$$

$$C_{D_x^4 \rho}^{(0), \text{CLBM1}} = (cs^2 \omega_3 - 12v_1^2 cs^2 \omega_3 - 2cs^2 + 3v_1^2 \omega_3 - 6v_1^2 + 2cs^4 - cs^4 \omega_3 + 6v_1^4 + 24v_1^2 cs^2 - 3v_1^4 \omega_3) \frac{1}{24\omega_3}$$

coefficient $C_{D_x^4 v_1}^{(0)}$ at $\frac{\partial^4 v_1}{\partial x_1^4}$:

$$C_{D_x^4 v_1}^{(0), \text{SRT}} = (-4 + 2\omega + 10v_1^2 - 3\omega cs^2 - 5v_1^2\omega + 6cs^2) \frac{\rho v_1}{12\omega}$$

$$C_{D_x^4 v_1}^{(0), \text{MRT1}} = (-4 - 3\omega_3 cs^2 - 5v_1^2\omega_3 + 10v_1^2 + 2\omega_3 + 6cs^2) \frac{\rho v_1}{12\omega_3}$$

$$C_{D_x^4 v_1}^{(0), \text{CLBM1}} = (-4 - 3cs^2\omega_3 + 6cs^2 - 5v_1^2\omega_3 + 10v_1^2 + 2\omega_3) \frac{\rho v_1}{12\omega_3}$$

3.2 Conservation of momentum equation

$$v_1 \frac{\partial \rho}{\partial t} + \rho \frac{\partial v_1}{\partial t} + C_{D_x \rho}^{(1)} \frac{\delta_l}{\delta_t} \frac{\partial \rho}{\partial x_1} + 2\rho v_1 \frac{\delta_l}{\delta_t} \frac{\partial v_1}{\partial x_1} + C_{D_x \rho, D_x v_1}^{(1)} \frac{\delta_l^2}{\delta_t} \frac{\partial \rho}{\partial x_1} \frac{\partial v_1}{\partial x_1} + C_{D_x v_1, D_x v_1}^{(1)} \frac{\delta_l^2}{\delta_t} \left(\frac{\partial v_1}{\partial x_1} \right)^2 + C_{D_x^2 \rho}^{(1)} \frac{\delta_l^2}{\delta_t} \frac{\partial^2 \rho}{\partial x_1^2} + C_{D_x^3 \rho}^{(1)} \frac{\delta_l^3}{\delta_t} \frac{\partial^3 \rho}{\partial x_1^3} + C_{D_x^3 v_1}^{(1)} \frac{\delta_l^3}{\delta_t} \frac{\partial^3 v_1}{\partial x_1^3} + C_{D_x^4 \rho}^{(1)} \frac{\delta_l^4}{\delta_t} \frac{\partial^4 \rho}{\partial x_1^4} + C_{D_x^4 v_1}^{(1)} \frac{\delta_l^4}{\delta_t} \frac{\partial^4 v_1}{\partial x_1^4} = 0,$$

where:

coefficient $C_{D_x \rho}^{(1)}$ **at** $\frac{\partial \rho}{\partial x_1}$:

$$C_{D_x \rho}^{(1), \text{SRT}} = (v_1^2 + cs^2)$$

$$C_{D_x \rho}^{(1), \text{MRT1}} = (cs^2 + v_1^2)$$

$$C_{D_x \rho}^{(1), \text{CLBM1}} = (cs^2 + v_1^2)$$

coefficient $C_{D_x \rho, D_x v_1}^{(1)}$ **at** $\frac{\partial \rho}{\partial x_1} \frac{\partial v_1}{\partial x_1}$:

$$C_{D_x \rho, D_x v_1}^{(1), \text{SRT}} = (-2 + \omega - 2\omega cs^2 + 6v_1^2 - 3v_1^2\omega + 4cs^2) \frac{1}{\omega}$$

$$C_{D_x \rho, D_x v_1}^{(1), \text{MRT1}} = (-2 - 2cs^2\omega_3 - 3v_1^2\omega_3 + 4cs^2 + 6v_1^2 + \omega_3) \frac{1}{\omega_3}$$

$$C_{D_x \rho, D_x v_1}^{(1), \text{CLBM1}} = (-2 + 4cs^2 - 3v_1^2\omega_3 + 6v_1^2 + \omega_3 - 2cs^2\omega_3) \frac{1}{\omega_3}$$

coefficient $C_{D_x v_1, D_x v_1}^{(1)}$ **at** $\left(\frac{\partial v_1}{\partial x_1} \right)^2$:

$$C_{D_x v_1, D_x v_1}^{(1), \text{SRT}} = (2 - \omega) \frac{3\rho v_1}{\omega}$$

$$C_{D_x v_1, D_x v_1}^{(1), \text{MRT1}} = (2 - \omega_3) \frac{3\rho v_1}{\omega_3}$$

$$C_{D_x v_1, D_x v_1}^{(1), \text{CLBM1}} = C_{D_x v_1, D_x v_1}^{(1), \text{MRT1}}$$

coefficient $C_{D_x^2 \rho}^{(1)}$ **at** $\frac{\partial^2 \rho}{\partial x_1^2}$:

$$C_{D_x^2 \rho}^{(1), \text{SRT}} = (-2 + \omega - 3\omega cs^2 + 2v_1^2 - v_1^2\omega + 6cs^2) \frac{v_1}{2\omega}$$

$$C_{D_x^2 \rho}^{(1), \text{MRT1}} = (-2 - 3cs^2\omega_3 - v_1^2\omega_3 + 6cs^2 + 2v_1^2 + \omega_3) \frac{v_1}{2\omega_3}$$

$$C_{D_x^2 \rho}^{(1), \text{CLBM1}} = (-2 + 6cs^2 - v_1^2\omega_3 + 2v_1^2 + \omega_3 - 3cs^2\omega_3) \frac{v_1}{2\omega_3}$$

coefficient $C_{D_x^2 v_1}^{(1)}$ **at** $\frac{\partial^2 v_1}{\partial x_1^2}$:

$$C_{D_x^2 v_1}^{(1), \text{SRT}} = (-2 + \omega - \omega cs^2 + 6v_1^2 - 3v_1^2\omega + 2cs^2) \frac{\rho}{2\omega}$$

$$C_{D_x^2 v_1}^{(1), \text{MRT1}} = (-2 - cs^2\omega_3 - 3v_1^2\omega_3 + 2cs^2 + 6v_1^2 + \omega_3) \frac{\rho}{2\omega_3}$$

$$C_{D_x^2 v_1}^{(1), \text{CLBM1}} = (-2 + 2cs^2 - 3v_1^2\omega_3 + 6v_1^2 + \omega_3 - cs^2\omega_3) \frac{\rho}{2\omega_3}$$

coefficient $C_{D_{x\rho}^3}^{(1)}$ **at** $\frac{\partial^3 \rho}{\partial x_1^3}$:

$$C_{D_{x\rho}^3}^{(1),\text{SRT}} = (-7v_1^2\omega^2 + 12cs^4 + 12\omega cs^2 - 36v_1^2 + \omega^2 cs^4 - 144v_1^2\omega cs^2 + 144v_1^2cs^2 + 36v_1^2\omega - 12\omega cs^4 + 24v_1^2\omega^2 cs^2 + 7v_1^4\omega^2 + 36v_1^4 - \omega^2 cs^2 - 36v_1^4\omega - 12cs^2) \frac{1}{12\omega^2}$$

$$C_{D_{x\rho}^3}^{(1),\text{MRT1}} = (-144v_1^2cs^2\omega_3 + 12cs^2\omega_3 - 7v_1^2\omega_3^2 + 36v_1^2\omega_3 - 12cs^2 - cs^2\omega_3^2 - 36v_1^2 + 24v_1^2cs^2\omega_3^2 + 12cs^4 + 7v_1^4\omega_3^2 - 12cs^4\omega_3 + 36v_1^4 + 144v_1^2cs^2 + cs^4\omega_3^2 - 36v_1^4\omega_3) \frac{1}{12\omega_3^2}$$

$$C_{D_{x\rho}^3}^{(1),\text{CLBM1}} = (-7v_1^2\omega_3^2 - 12cs^2 + cs^4\omega_3^2 - 12cs^4\omega_3 + 36v_1^2\omega_3 - 36v_1^2 + 7v_1^4\omega_3^2 - cs^2\omega_3^2 + 36v_1^4 + 144v_1^2cs^2 + 24v_1^2cs^2\omega_3^2 - 144v_1^2cs^2\omega_3 + 12cs^4 + 12cs^2\omega_3 - 36v_1^4\omega_3) \frac{1}{12\omega_3^2}$$

coefficient $C_{D_{xv_1}^3}^{(1)}$ **at** $\frac{\partial^3 v_1}{\partial x_1^3}$:

$$C_{D_{xv_1}^3}^{(1),\text{SRT}} = (-24 + 11v_1^2\omega^2 + 24\omega - 36\omega cs^2 + 60v_1^2 - 60v_1^2\omega - 4\omega^2 + 5\omega^2 cs^2 + 36cs^2) \frac{\rho v_1}{6\omega^{\frac{1}{2}}}$$

$$C_{D_{xv_1}^3}^{(1),\text{MRT1}} = (-24 - 36cs^2\omega_3 + 11v_1^2\omega_3^2 - 60v_1^2\omega_3 + 36cs^2 + 5cs^2\omega_3^2 + 60v_1^2 - 4\omega_3^2 + 24\omega_3) \frac{\rho v_1}{6\omega_3^{\frac{1}{2}}}$$

$$C_{D_{xv_1}^3}^{(1),\text{CLBM1}} = (-24 + 11v_1^2\omega_3^2 + 36cs^2 - 60v_1^2\omega_3 + 60v_1^2 + 5cs^2\omega_3^2 - 4\omega_3^2 + 24\omega_3 - 36cs^2\omega_3) \frac{\rho v_1}{6\omega_3^{\frac{1}{2}}}$$

coefficient $C_{D_{x\rho}^4}^{(1)}$ **at** $\frac{\partial^4 \rho}{\partial x_1^4}$:

$$C_{D_{x\rho}^4}^{(1),\text{SRT}} = (12 - 98v_1^2\omega^2 + 10v_1^2\omega^3 + 144cs^4 - 5\omega^3 cs^4 - 18\omega + 198\omega cs^2 - 156v_1^2 + 82\omega^2 cs^4 - 1008v_1^2\omega cs^2 + 672v_1^2cs^2 + 234v_1^2\omega - 9v_1^4\omega^3 + 8\omega^2 - 216\omega cs^4 + 404v_1^2\omega^2 cs^2 - \omega^3 + 90v_1^4\omega^2 + 144v_1^4 - 78\omega^2 cs^2 - 34v_1^2\omega^3 cs^2 - 216v_1^4\omega + 6\omega^3 cs^2 - 132cs^2) \frac{v_1}{12\omega^{\frac{3}{2}}}$$

$$C_{D_{x\rho}^4}^{(1),\text{MRT1}} = (12 - 1008v_1^2cs^2\omega_3 + 10v_1^2\omega_3^3 + 198cs^2\omega_3 - 98v_1^2\omega_3^2 - 34v_1^2cs^2\omega_3^3 + 234v_1^2\omega_3 - 132cs^2 - 78cs^2\omega_3^2 - 156v_1^2 + 404v_1^2cs^2\omega_3^2 + 6cs^2\omega_3^3 + 144cs^4 + 90v_1^4\omega_3^2 - 216cs^4\omega_3 - \omega_3^3 + 8\omega_3^2 - 9v_1^4\omega_3^3 + 144v_1^4 - 18\omega_3 + 672v_1^2cs^2 - 5cs^4\omega_3^3 + 82cs^4\omega_3^2 - 216v_1^4\omega_3) \frac{v_1}{12\omega_3^{\frac{3}{2}}}$$

$$C_{D_{x\rho}^4}^{(1),\text{CLBM1}} = (12 + 10v_1^2\omega_3^3 - 5cs^4\omega_3^3 - 98v_1^2\omega_3^2 - 132cs^2 + 82cs^4\omega_3^2 - 216cs^4\omega_3 + 234v_1^2\omega_3 - 156v_1^2 + 90v_1^4\omega_3^2 - \omega_3^3 - 78cs^2\omega_3^2 - 34v_1^2cs^2\omega_3^3 + 8\omega_3^2 - 9v_1^4\omega_3^3 + 144v_1^4 + 672v_1^2cs^2 + 6cs^2\omega_3^3 + 404v_1^2cs^2\omega_3^2 - 18\omega_3 - 1008v_1^2cs^2\omega_3 + 144cs^4 + 198cs^2\omega_3 - 216v_1^4\omega_3) \frac{v_1}{12\omega_3^{\frac{3}{2}}}$$

coefficient $C_{D_{xv_1}^4}^{(1)}$ **at** $\frac{\partial^4 v_1}{\partial x_1^4}$:

$$C_{D_{xv_1}^4}^{(1),\text{SRT}} = (12 - 154v_1^2\omega^2 + 14v_1^2\omega^3 + 24cs^4 - \omega^3 cs^4 - 18\omega + 54\omega cs^2 - 252v_1^2 + 14\omega^2 cs^4 - 648v_1^2\omega cs^2 + 432v_1^2cs^2 + 378v_1^2\omega - 29v_1^4\omega^3 + 8\omega^2 - 36\omega cs^4 + 252v_1^2\omega^2 cs^2 - \omega^3 + 310v_1^4\omega^2 + 504v_1^4 - 22\omega^2 cs^2 - 18v_1^2\omega^3 cs^2 - 756v_1^4\omega + 2\omega^3 cs^2 - 36cs^2) \frac{\rho}{12\omega^{\frac{3}{2}}}$$

$$C_{D_{xv_1}^4}^{(1),\text{MRT1}} = (12 - 648v_1^2cs^2\omega_3 + 14v_1^2\omega_3^3 + 54cs^2\omega_3 - 154v_1^2\omega_3^2 - 18v_1^2cs^2\omega_3^3 + 378v_1^2\omega_3 - 36cs^2 - 22cs^2\omega_3^2 - 252v_1^2 + 252v_1^2cs^2\omega_3^2 + 2cs^2\omega_3^3 + 24cs^4 + 310v_1^4\omega_3^2 - 36cs^4\omega_3 - \omega_3^3 + 8\omega_3^2 - 29v_1^4\omega_3^3 + 504v_1^4 - 18\omega_3 + 432v_1^2cs^2 - cs^4\omega_3^3 + 14cs^4\omega_3^2 - 756v_1^4\omega_3) \frac{\rho}{12\omega_3^{\frac{3}{2}}}$$

$$C_{D_{xv_1}^4}^{(1),\text{CLBM1}} = (12 + 14v_1^2\omega_3^3 - cs^4\omega_3^3 - 154v_1^2\omega_3^2 - 36cs^2 + 14cs^4\omega_3^2 - 36cs^4\omega_3 + 378v_1^2\omega_3 - 252v_1^2 + 310v_1^4\omega_3^2 - \omega_3^3 - 22cs^2\omega_3^2 - 18v_1^2cs^2\omega_3^3 + 8\omega_3^2 - 29v_1^4\omega_3^3 + 504v_1^4 + 432v_1^2cs^2 + 2cs^2\omega_3^3 + 252v_1^2cs^2\omega_3^2 - 18\omega_3 - 648v_1^2cs^2\omega_3 + 24cs^4 + 54cs^2\omega_3 - 756v_1^4\omega_3) \frac{\rho}{12\omega_3^{\frac{3}{2}}}$$

References

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